Since there are six poles associated with four design positions (P_{12}, P_{13}, P_{14}, P_{23}, P_{24}, P_{34}) it follows that there are three *Opposite Pole Pairs* among those six poles.

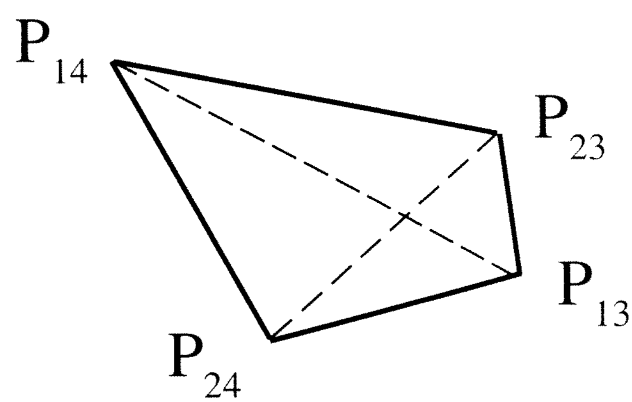
P_{12}—P_{34}
P_{23}—P_{14}
P_{24}—P_{12}

☜ All the *Opposite Pole Pairs* for Four Positions

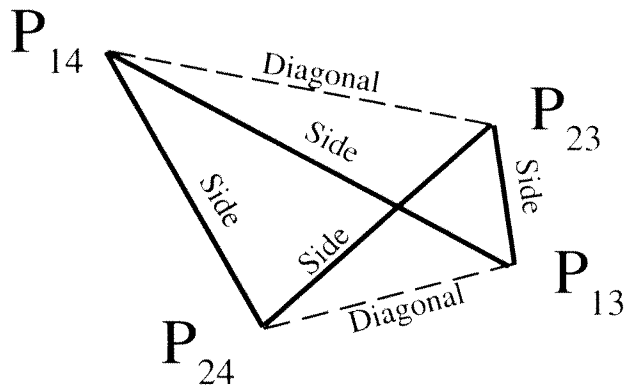
If you take any two sets of these *Opposite Pole Pairs*, you can use the four corresponding poles as the four vertices of a quadrilateral. Big deal, you say. If you use the sets of *Opposite Pole Pairs* as the *diagonals* of your quadrilateral, it is a very big deal, because then you have what is known to the cognoscenti in the trade as an *Opposite Pole Pair Quadrilateral!*

An *Opposite Pole Pair Quadrilateral* is one that has an *Opposite Pole Pair* for each of its diagonals. How do I know? By definition! That's just how it is! This is a mathematical definition and not something based on common sense. (Well-known fact among third graders: Common sense has no place in mathematics.)

Usually if you saw the figure at the right and were told the four vertices defined a quadrilatral you would say that the diagonals were the lines from P_{14} to P_{13} and from P_{23} to P_{24}. In this case, YOU WOULD BE WRONG!

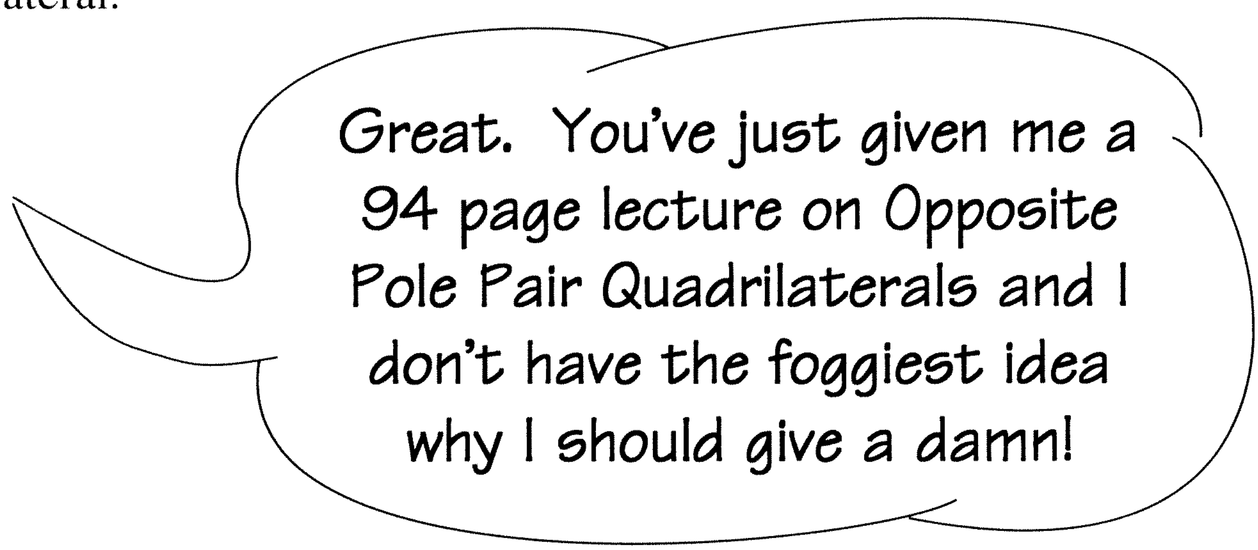


If this is an *Opposite Pole Pair Quadrilateral* then, by definition, the *diagonals* are the lines from P_{14} to P_{23} and from P_{24} to P_{13}.



The *sides* of the *O.P.P.Q. (Opposite Pole Pair Quadrilateral,* for those who hate getting the abbreviated version of the story) are the lines connecting poles with a common subscript. For instance, the line P_{14}—P_{13} is a *side* of this *O.P.P.Q.*, since it has a 1 subscript in common.

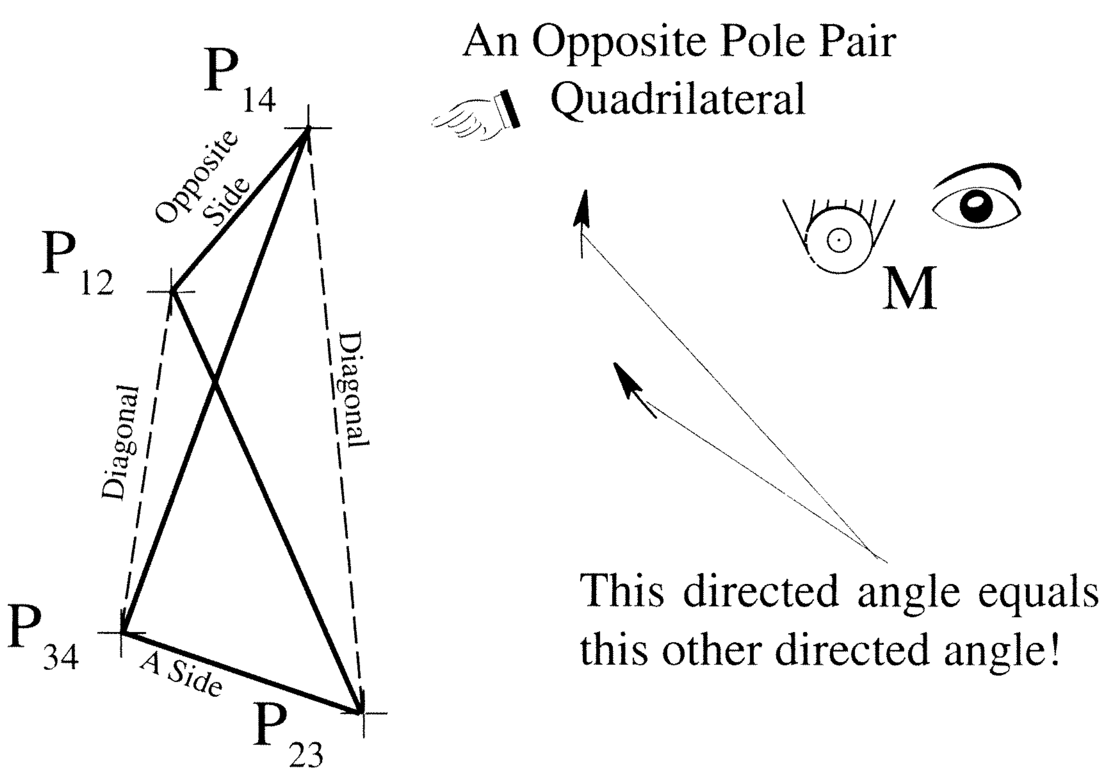
O.K. Now you know what an *Opposite Pole Pair Quadrilateral* is. Since there are three sets of *Opposite Pole Pairs* associated with four positions and since it takes two diagonals to define a quadrilateral, it follows that there are three possible *O.P.P.Q.*'s you can form— you can use any two *Opposite Pole Pairs* as diagonals for your quadrilateral.



The reason you should give a damn about *Opposite Pole Pair Quadrilaterals* is that

> YOU KNOW YOU ARE STANDING ON A CENTERPOINT FOR FOUR POSITIONS IF YOU SEE OPPOSITE SIDES OF AN OPPOSITE POLE PAIR QUADRILATERAL UNDER EQUAL ANGLES!

There. I've said it. Are you happy now? In other words, a *general* way to know you are standing on a centerpoint is if you are standing in a place where you see opposite sides of any *O.P.P.Q.* under equal angles (or perhaps angles that differ by 180°), as shown below.



"There's gotta be an easier way to find these centerpoints..." (you say)

"There is!" (I say)
(I just needed to drag out the story a bit to make this tome weighty enough so my department chairman would think I had been doing something useful with my time instead of just picking belly button lint and weaving it into cloth!)

Coming up is a graphical procedure by which we can systematically *construct* the locations of points that meet the conditions we now know centerpoints must satisfy— no random searching around required! It's a closed-form graphical procedure that at least stands a *chance* of finding the desired points without a lot of wasted effort.

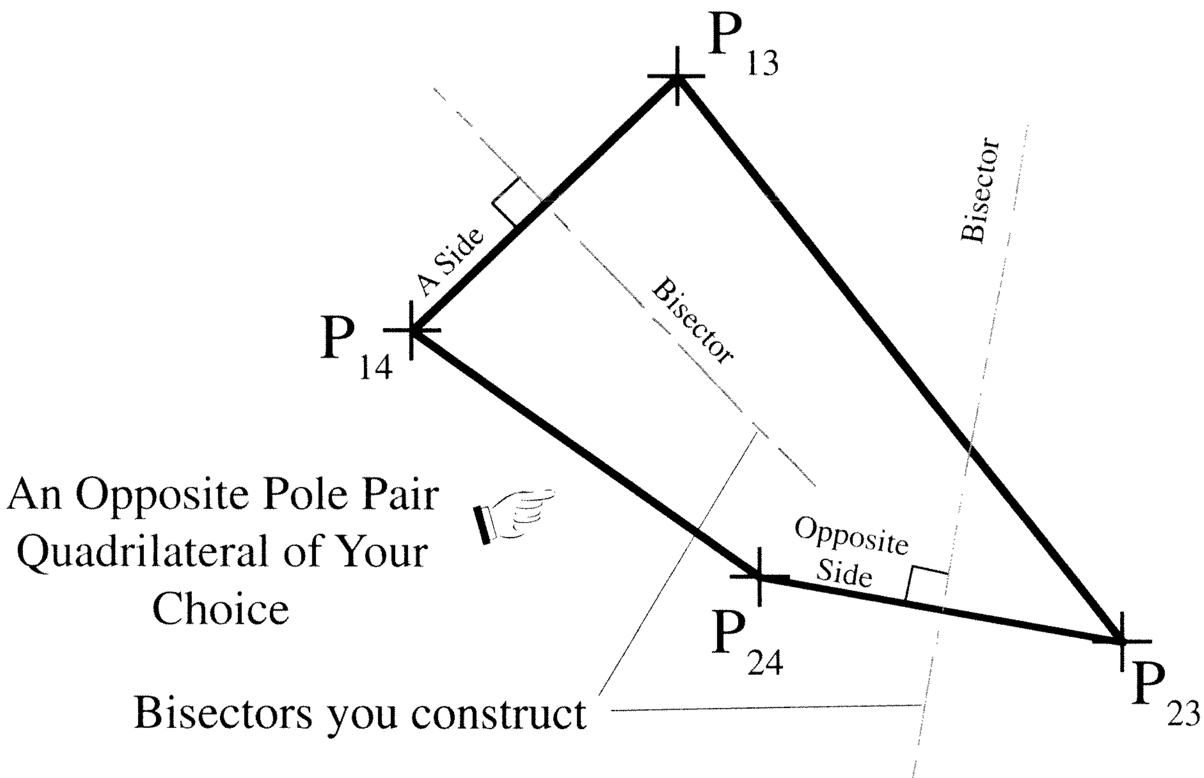
Here's how it works:

☞ First, find the six poles corresponding to the four given design positions.

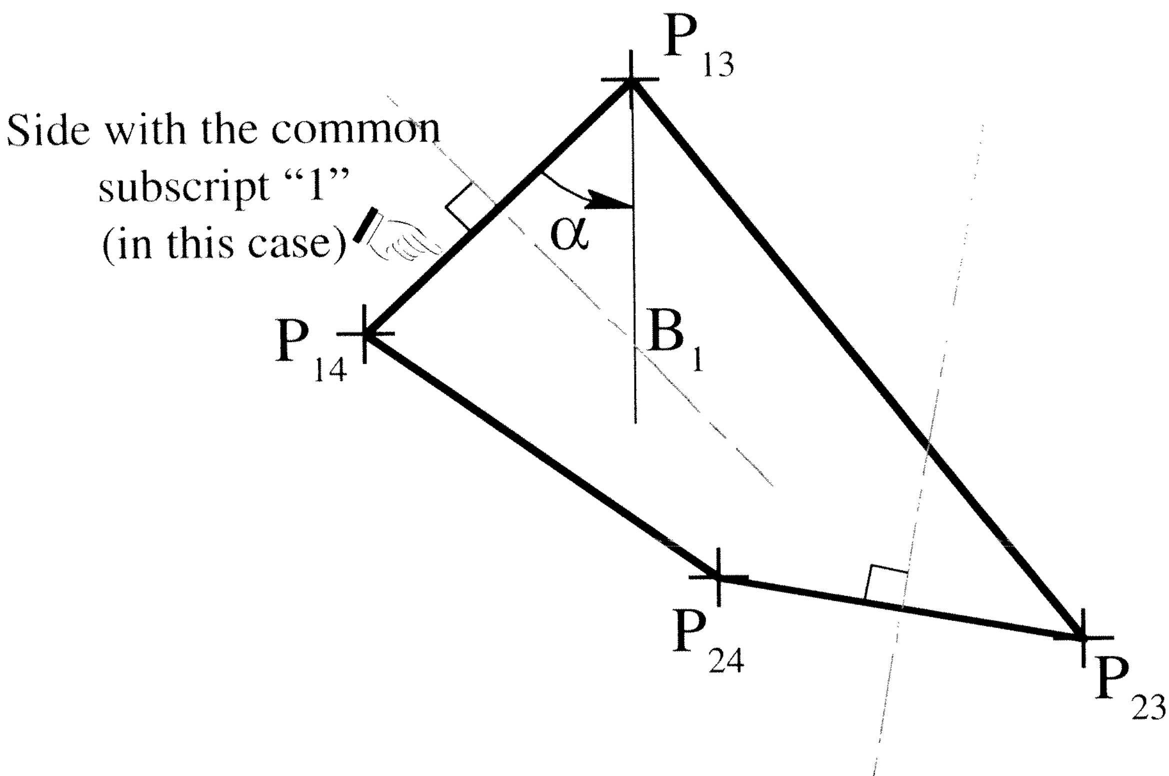
☞ Then, pick out two convenient Opposite Pole Pairs to serve as diagonals for an Opposite Pole Pair Quadrilateral.

☞ Next, arbitrarily pick any two opposite sides of the selected quadrilateral.

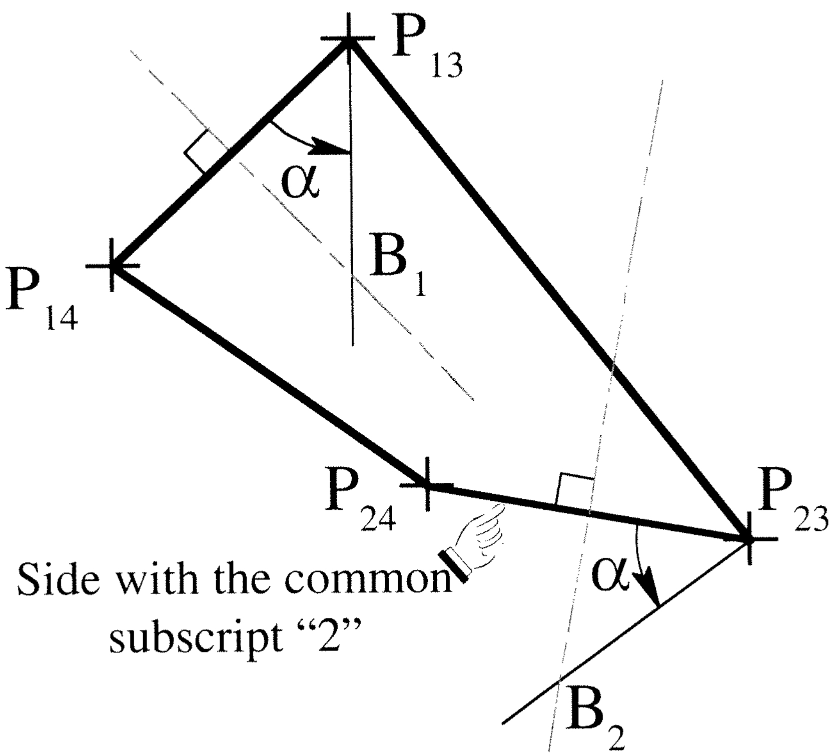
☞ Now, construct the perpendicular bisectors to both the sides you selected.



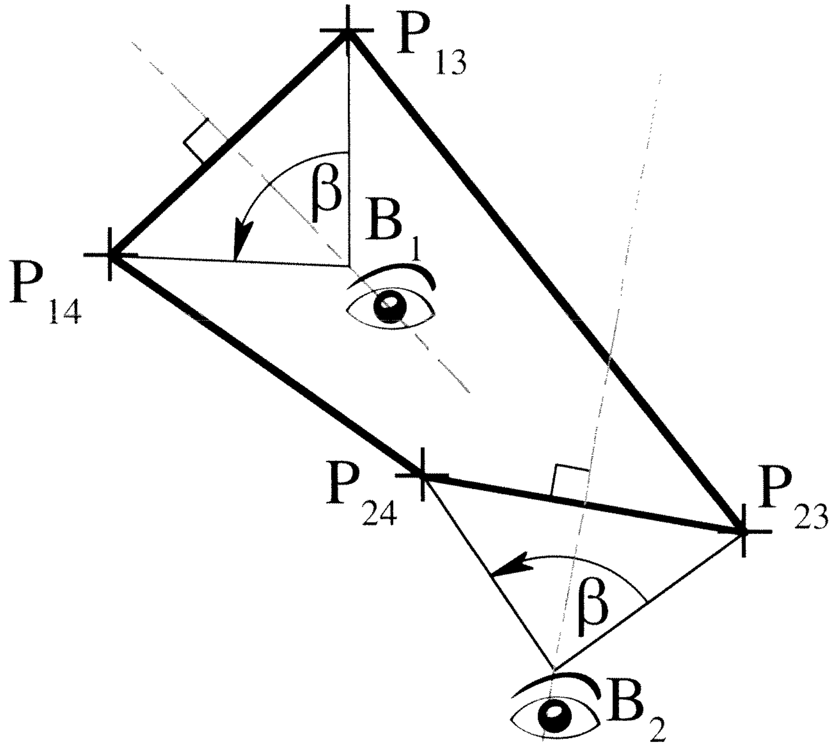
☞ From the pole at the end of one of the sides lay out an arbitrary signed angle α as shown on the next figure. Mark the point where it cuts the bisector to that side. (Since I laid out the angle on the side with a #1 subscript in common, I am labeling this point B_1 so as not to get confused.) (B as in "Bisector" and 1 as in "1", get it?)



Lay out the *same* signed angle α on the opposite side. Again mark the point where it cuts the bisector.



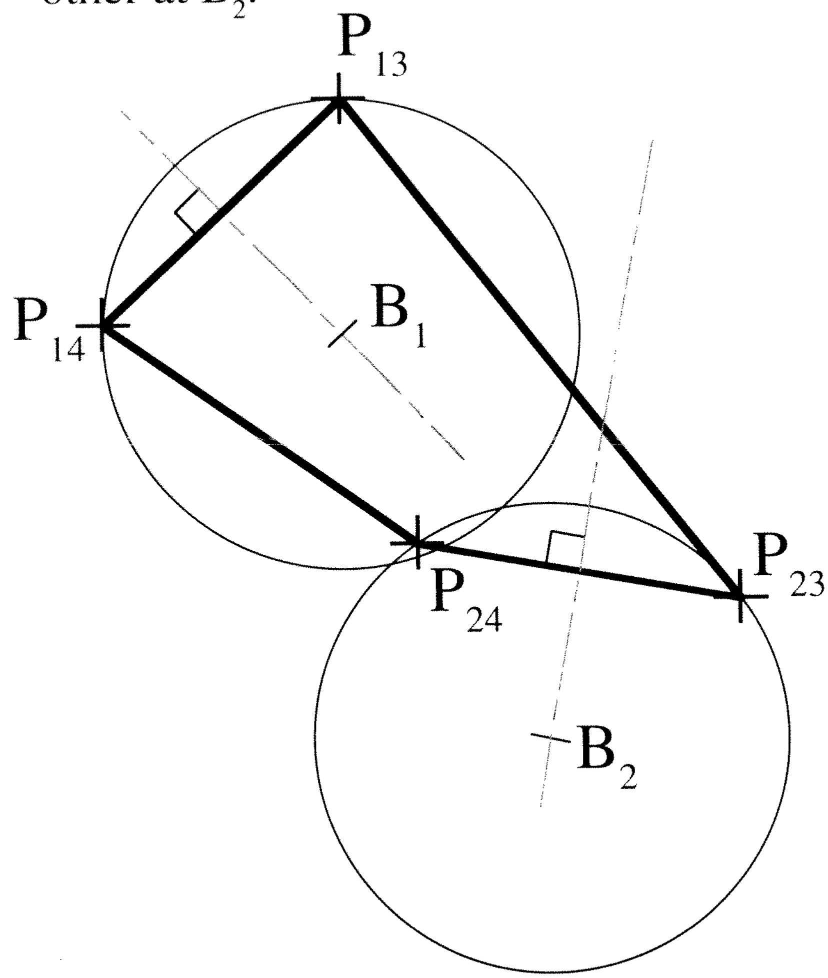
☞ Since, in this case, I am now working on the side with a #2 subscript in common, I am labeling this point B_2 'cause that somehow seems more logical than naming it "Gwendolyn".



☞ Put one of your eyeballs at B_1 and the other at B_2. (I know it hurts like hell but just do it!)

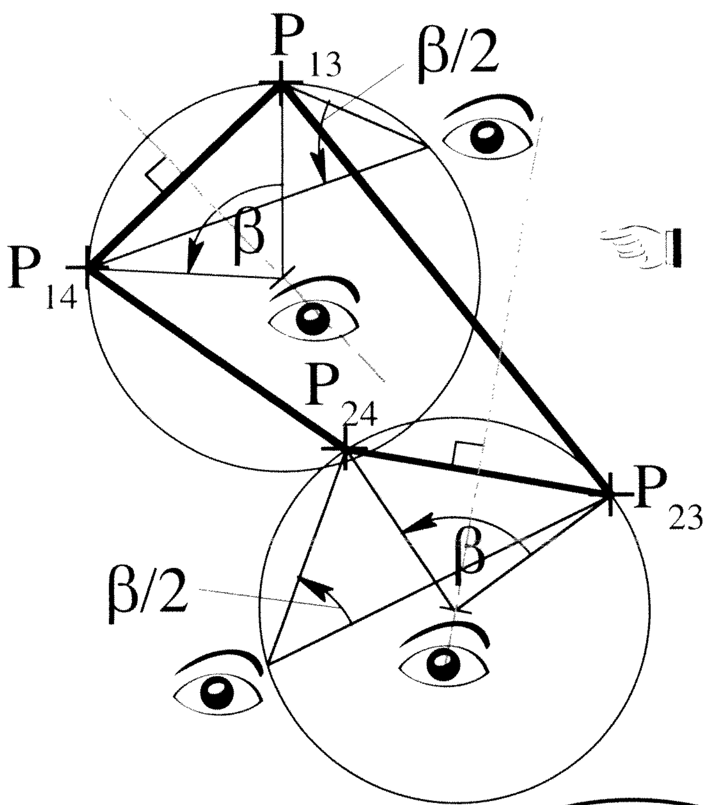
☞ Notice that the eyeball at B_2 sees the side from P_{23} to P_{24} under the *same signed angle* (both in magnitude and direction) as does your other eyeball when it looks from P_{13} to P_{14}. In other words, as you look from the 3 to the 4 pole you see them under the same angle regardless of which subscript is in common.

Now swing two circles, one centered at B_1 and the other at B_2.

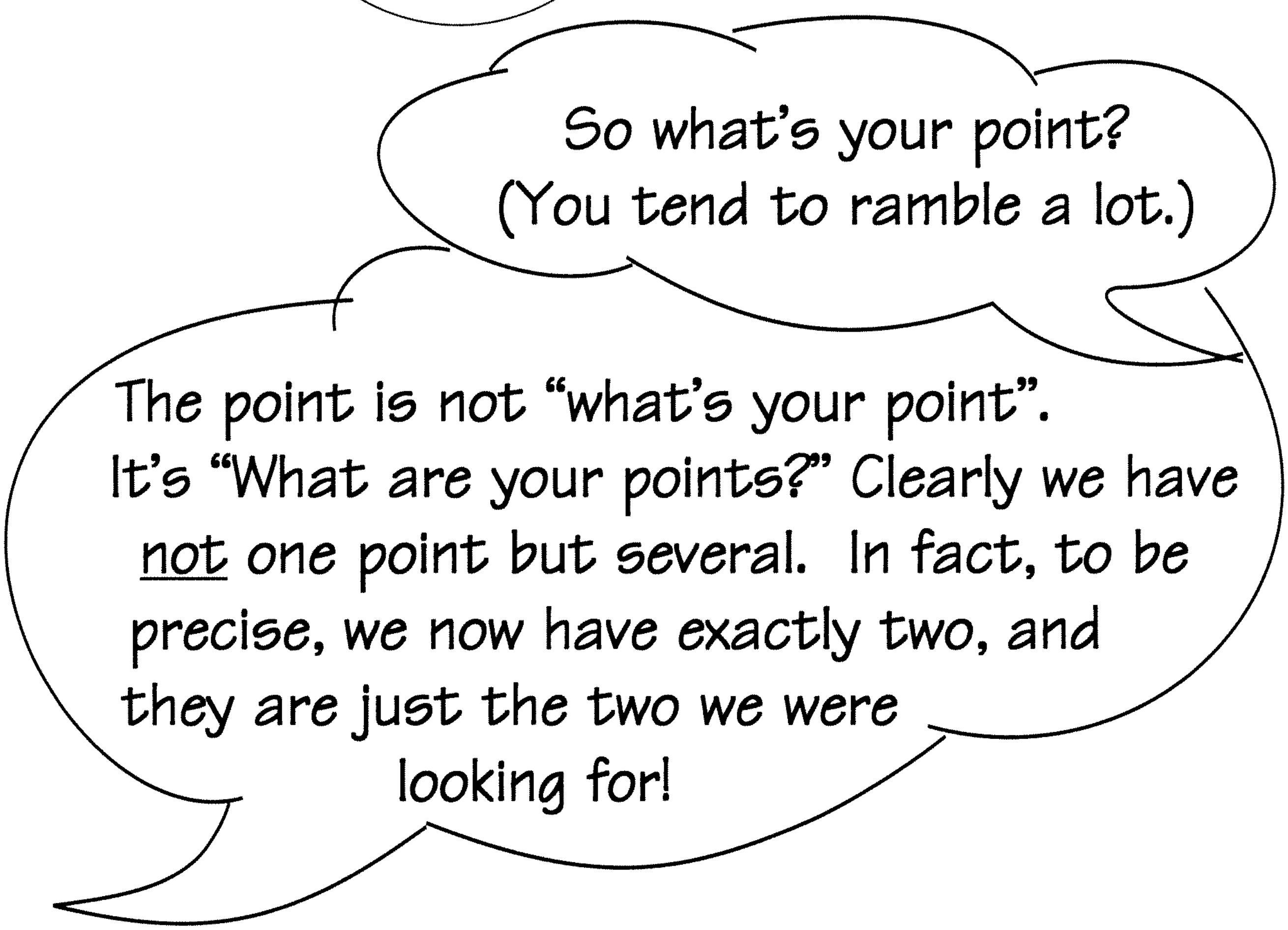


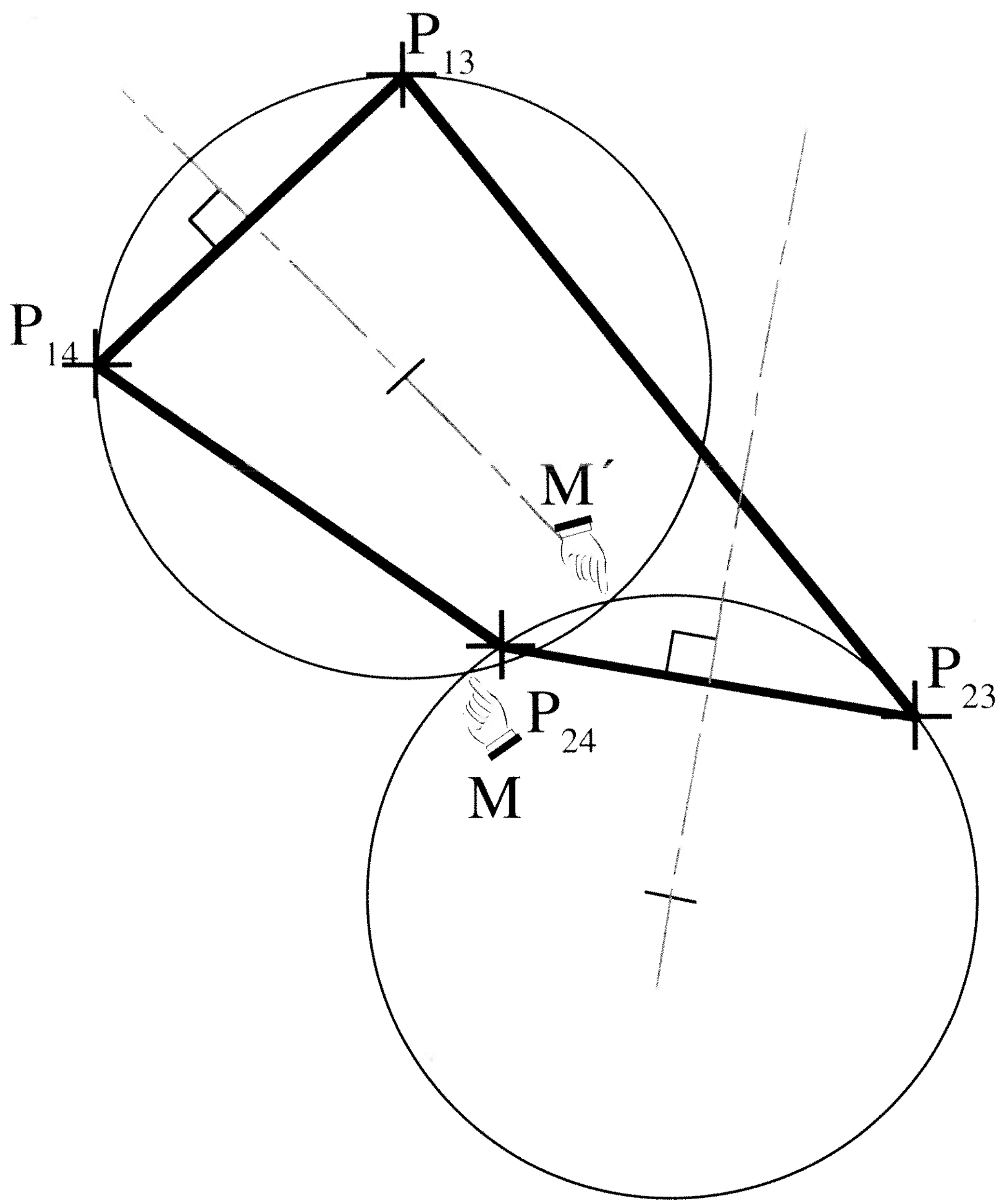
Way back in high school you probably learned that a point on a circle sees a chord of the circle under half the central angle. (Well. In *your* case you probably *didn't* learn it but you should have. In any event, it's true.)

Since the chords P_{13}-P_{14} and P_{23}-P_{24} subtend the same central angle β as seen from the centers of the two circles, they must subtend $\beta/2$ as seen from any point on the circles themselves.



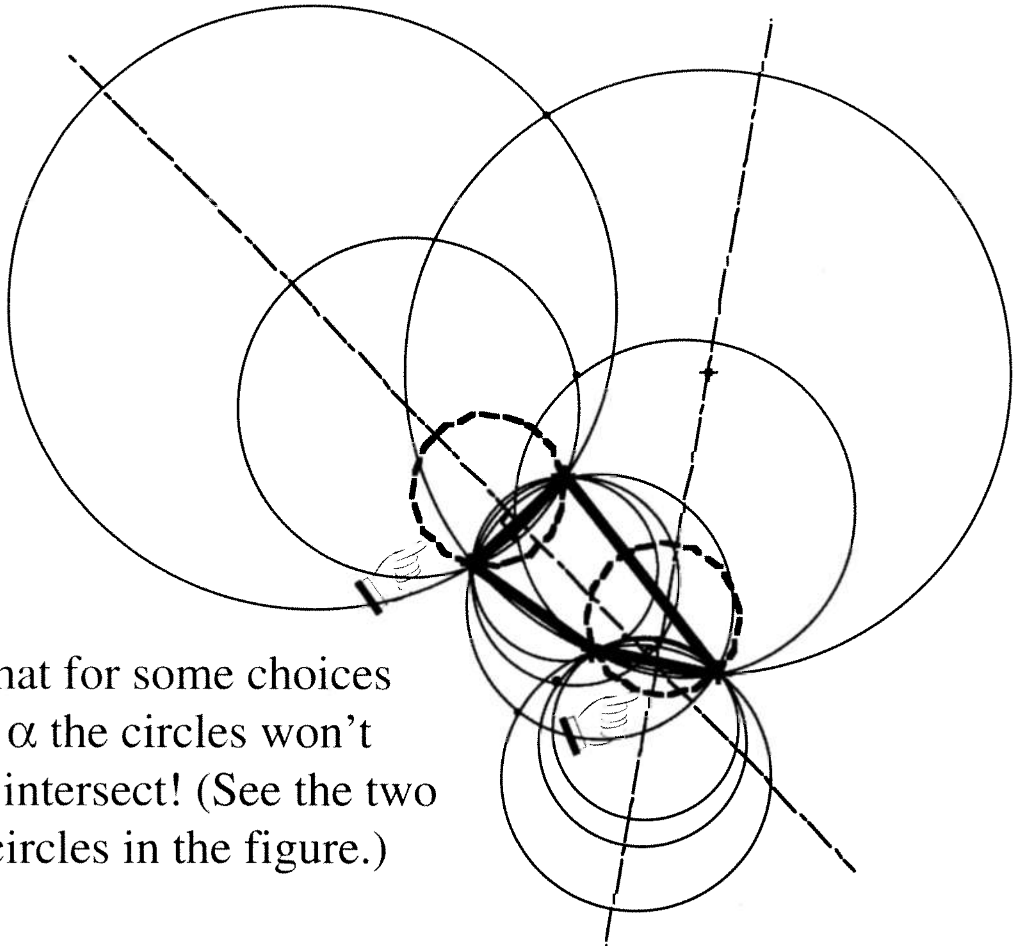
Looking from any point on the circumference of the first circle we see from P_{13} to P_{14} under the same signed angle $\beta/2$ as we see P_{23} to P_{24} looking from any point on the second circle.





We have just succeeded in constructing—count 'em— not one but two *centerpoints*, M and M´. The places where the two circles intersect see *opposite sides* of the *opposite pole pair quadrilateral* under equal angles! Call out the brass band! (Actually, the angles may not be equal but may differ by 180°from one another, but they stilll satisfy the requirements for valid centerpoints.)

Repeat the process. Just vary the angle α slightly, swing another set of circles, and mark the locations of the new intersections M and M´. Bingo! You've got two more centerpoints. Keep doing this an infinite number of times, varying the angle α from -180° to + 180° and you will easily generate an infinite number of little points like M and M´.

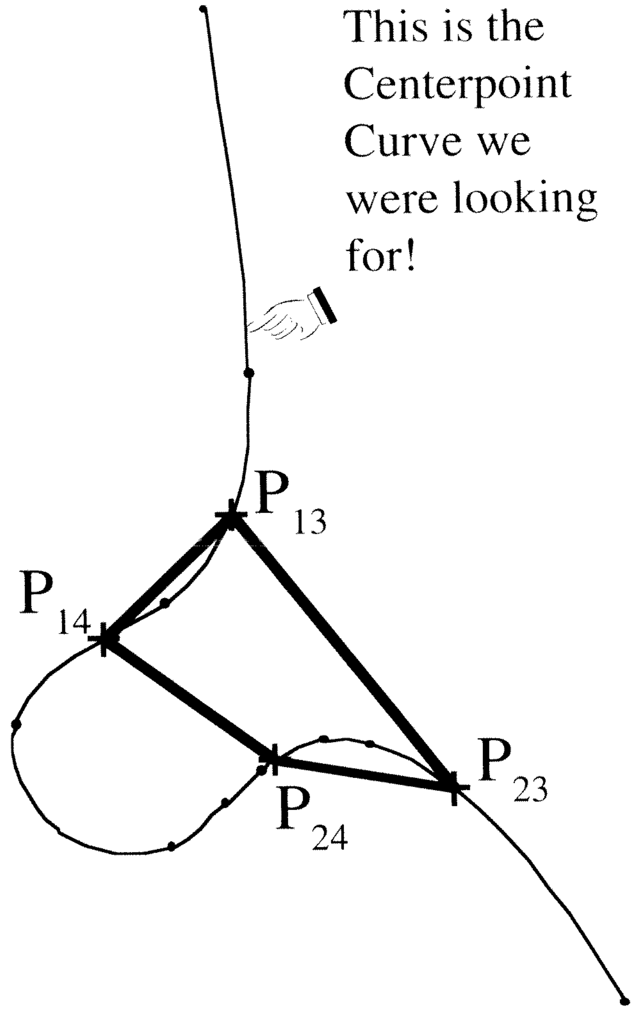


Notice that for some choices of angle α the circles won't actually intersect! (See the two dashed circles in the figure.)

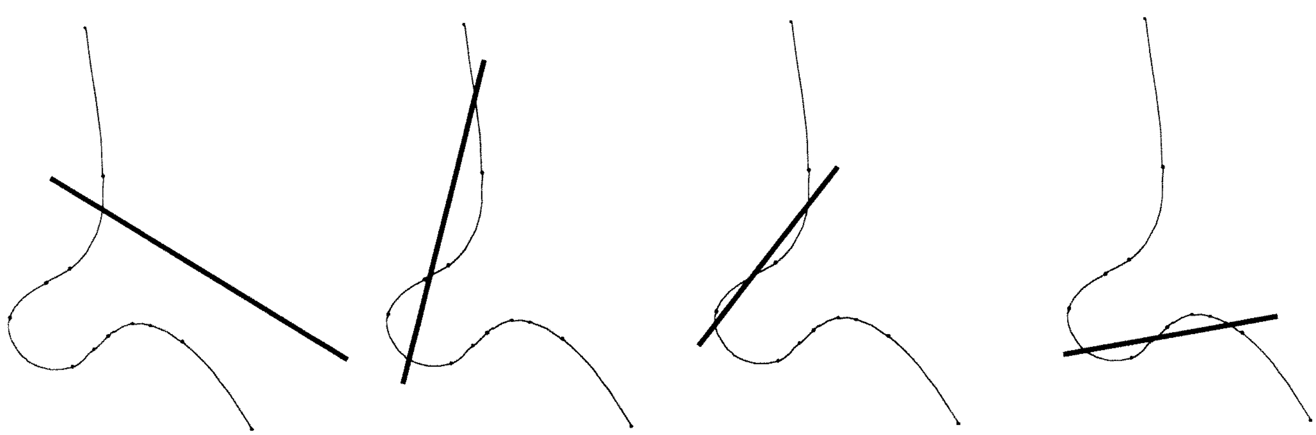
O.K. So I lied. It's not easy. Constructing an infinite number of points will take you an infinitely long time but what the hell, you've got nothing better to do with your weekends. It's not like you had a sex life or anything. Actually, you probably don't need all the points to get an adequate idea of what is going on so quit after the first month or so of graphical constructions, or if you get a date, whichever comes earlier.

Hook the M's and M´´s all together in order and you get a picture that looks kind of like a squirming octopus. (It's kind of like connecting the dots on the kiddy page in the Sunday Comics and getting a picture of Santa.)

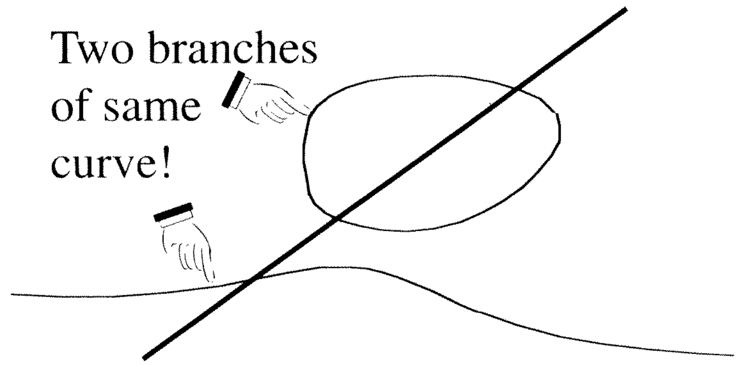
This is the Centerpoint Curve we were looking for!

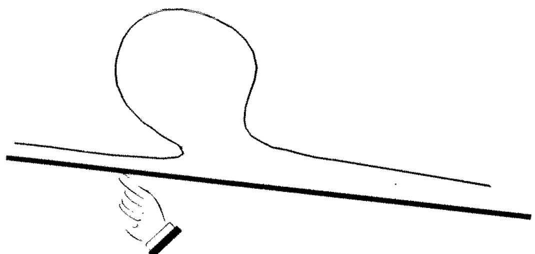


If you've done it right, the curve will probably look like an octopus because it is a cubic curve. Cubics are nice smooth octopus-like curves that have at most three intersections with a straight line. So, if you find your curve wiggles back and forth a lot it means you have botched the job up. Instead of being octopus-like your curve is all squiddley. The straight-line rule provides a quick graphical check to see if things seem to be going along O.K..



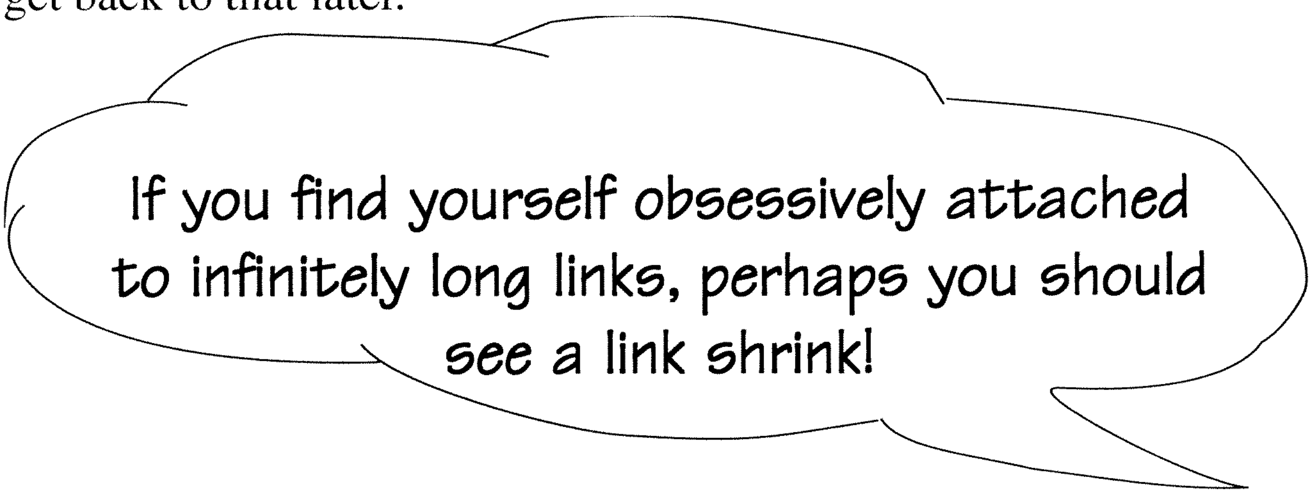
Even though the curves usually look like octopi, they often break off like a droplet and have two branches as in this case. The straight line rule still works, as this is just a single cubic curve even though it has two parts.





(This isn't the whole asymptote. In fact it's less than a half-assymptote!)

The curve also has an asymptote and extends to infinity. For most engineering design purposes, you probably don't need to construct the parts of the curve that lie beyond Vladavostok, but you should know that theoretically they exist. Actually, the points at infinity are very interesting, since they are centers for an infinitely long link. Nobody but the military can afford an infinitely long link but lots of folks are interested in sliders which behave like they were attached to an infinitely long link. We'll get back to that later.



So, Professor Kaufman. I understand you are working on the world's first biological mechanism made from the DNA of a prune?

That's true. You should see that linkage run!

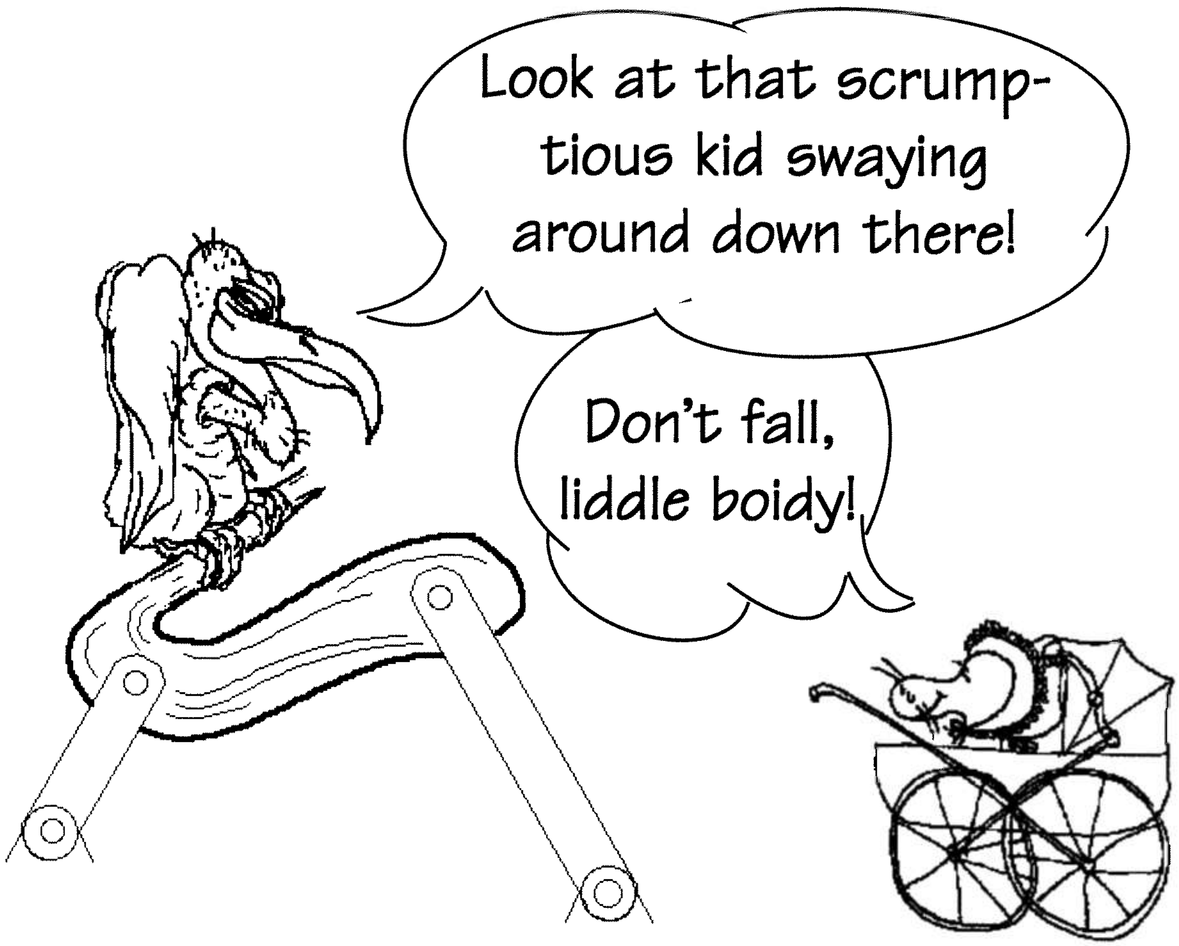
Now that we have a closed-form graphical procedure for con structing centerpoints, how can we construct the circlepoint curve, you ask.

Now that we have a closed-form graphical procedure for constructing centerpoints, how can we construct the circlepoint curve, I ask?

The answer is "With great arduous & laborious difficulty, slaving away for months at a drafting table, surrounded by reams of paper and constructions, floor littered with crumpled tracing paper overlays from fruitless hours applying obscure theorems from higher-level projective geometry !" (Well, there I go again, exaggerating for dramatic effect. It's not with great difficulty. It's about the same amount of difficulty as for the centerpoint curve". It's no piece of cake but it's not incredibly hard either.)

What's the difference between a circlepoint curve and a centerpoint curve? They're really just the same! It only depends on your point of view.

Suppose you had two observers, one riding on the moving coupler link of a four-bar and the other standing on the frame. The person riding on the moving body isn't aware that the coupler link is moving (until it swings around to the point where they start to fall off).



Each observer thinks everything atached to their own reference frame is stationary. Pivots attached to their own body appear as fixed centerpoints and pivots on the other person's body are seen as swinging on circular arcs centered at the fixed centerpoints on the body on which they are riding. Thus, one person's centerpoints are the other person's circlepoints and vice versa. If you don't like your vice versa, stay away from kinematic inversions.

In other words, the circlepoint curve is constructed just the way a centerpoint curve is drawn, but it is drawn on the moving body instead of the fixed body. To do that, you need to use properties of the moving poles instead of properties of the fixed poles.

Before, we drew the *centerpoint curve* by using an *opposite pole-pair quadrilateral.* If, instead, we went through exactly the same procedure, but using an *opposite image pole pair quadrilateral*, we would end up with a *circlepoint curve* drawn on the moving body in it's number one position (assuming all the image poles were in the number one positon). Why? By analogy. Because circlepoints are to centerpoints as image poles are to poles and as moving bodies are to fixed bodies; because university faculty is to university administrators as bird dung is to... well, you get the idea.


A few pairs of pretty Burmester Circlepoint and Centerpoint curves twistina in the wind...

O.K. At this point we have constructed a reasonable length of both the circlepoint and the centerpoint curves in the areas of interest, or so we hope.

What's "the area of interest"?

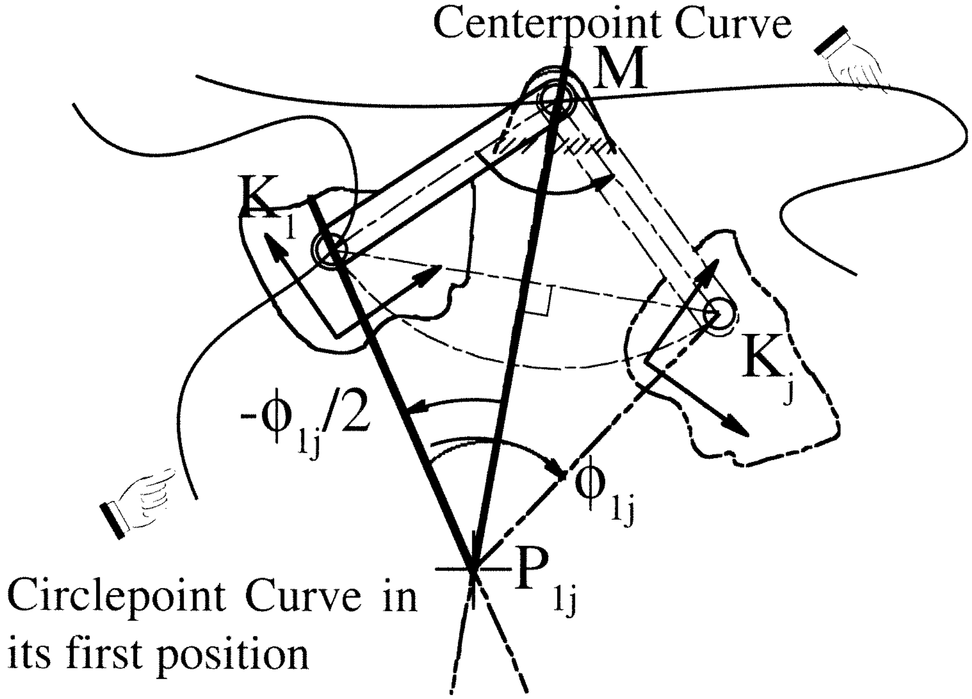
That's the general vicinity in which you want your mechanism pivots to lie.

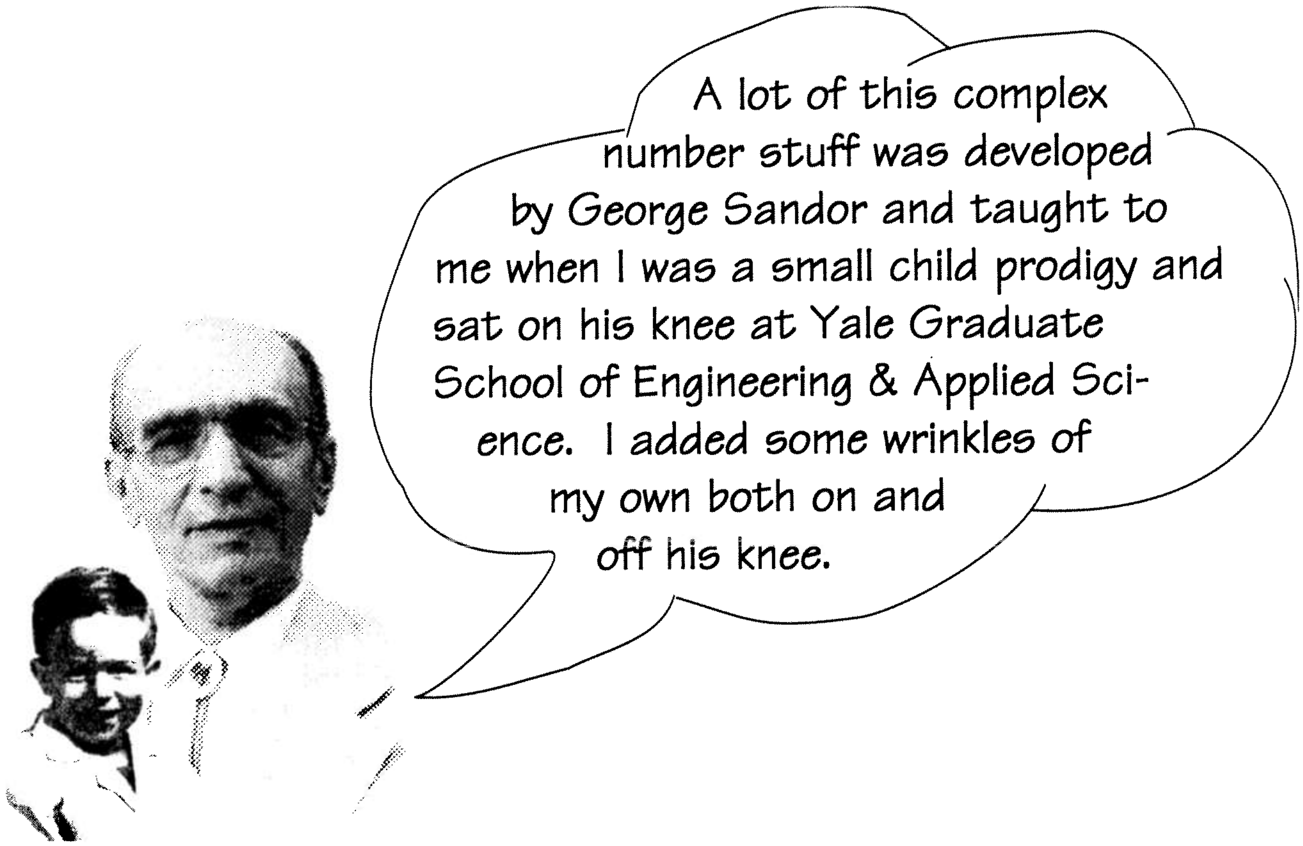
Why do you say "or so we hope"?

"So we hope" is an incantation we make because the curves almost never end up where you want them to. "So we hope" is a testament to the pathetic optimisim of the human spirit. It's another way of saying "a sucker is born every minute."

All that is left is for us to pick off a couple of matching sets of circlepoints and centerpoints. Each circlepoint corresponds to one and only one centerpoint and vice versa. Knowing either a circlepoint or a centerpoint, you should be able to find the other. But with all these damn points on the circlepoint and centerpoint curves, how do we match them up? Which circlepoints go with which centerpoints? Typically, we want to find a couple of matching sets to use as pivots for a four-bar linkage as seen in its number one position.

One way to match up corresponding points on the two curves is by using any pole with a number one subscript, such as P_{12}. You know that standing on the pole, you see the circlepoint and centerpoint under the angle $\phi_{1-2/2}$. Thus if you pick any point on either the circlepoint or the centerpoint curve, the corresponding point on the other curve will be on a line $\phi_{1-2/2}$ away. (You just need to be careful to keep the sign of the angle straight.) If there is any ambiguity, you can use another pole and it's half angle of rotation as a check. The right point will lie at the intersection of the two lines from the two poles.





The author sitting on the knee of his mentor, Dr. George N. Sandor and learning to play with complex numbers and make corny puns.

Given: n positions of plane π, specified in the fixed coordi nate system by means of the vectors

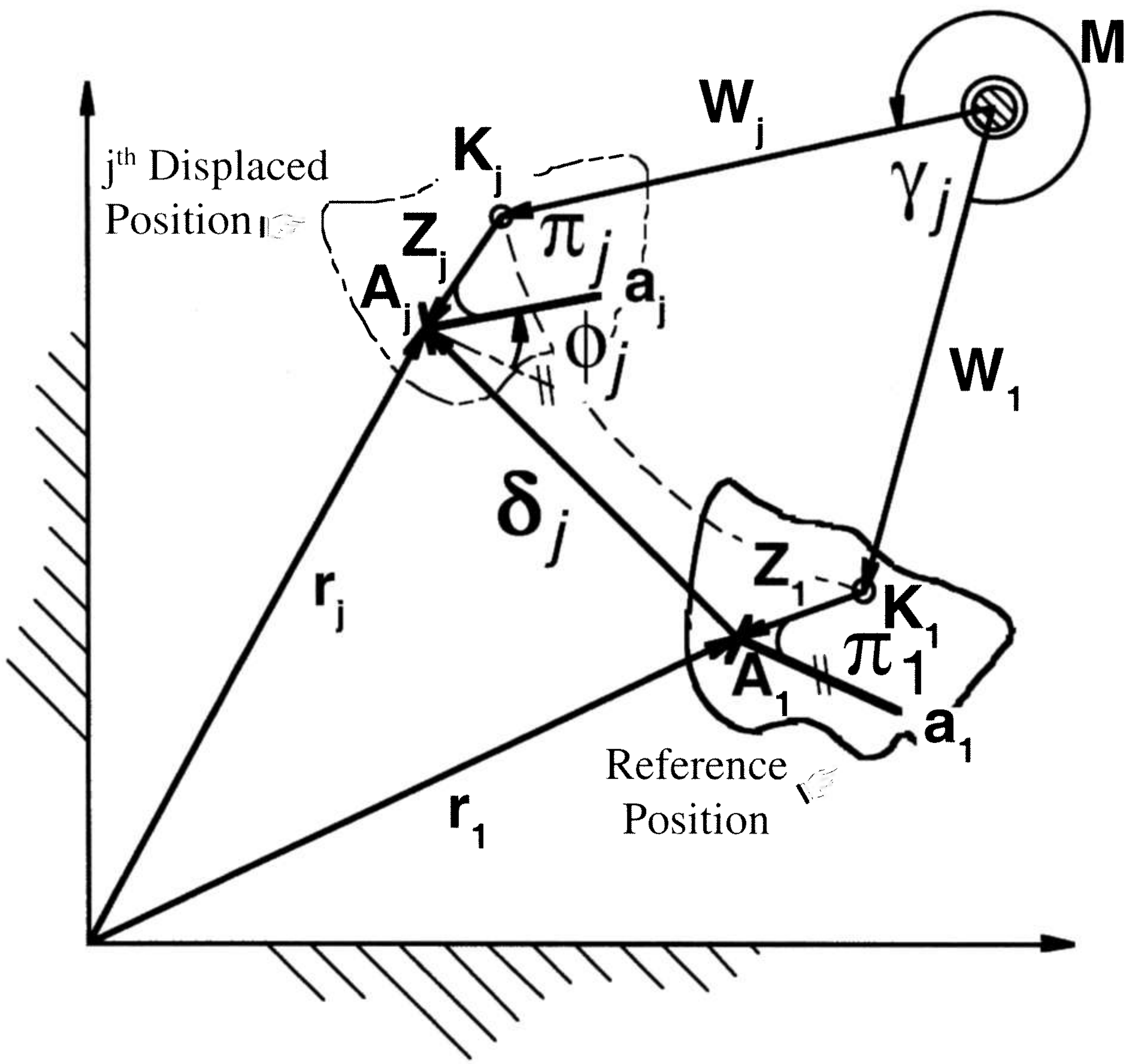
$$\mathbf{r}_j, \; j = 1,2,\ldots n$$

and the displacement angles

$$\phi_j, \; j = 2, 3,\ldots n$$

which indicate the rotation of the body relative to its orientation in the first (reference) position. (Any convenient point A can be used as a reference for the vectors $\mathbf{r}_j$.

Find: Circlepoints **K** and Centerpoints **M** corresponding to each circlepoint. (There is a one to one correspondence between the circlepoints and the centerpoints. Each circlepoint describes a circle about one center, unless you wear bifocals.)



In the reference position, the unknown circlepoint, K_1 bears a certain unknown relationship to the known point A_1. Let the unknown vector $\mathbf{Z}_1$, <u>*rigidly attached to the moving plane*</u>, locate point A_1 with respect to K_1.

Let $\mathbf{W}_1$ locate the unknown circlepoint K_1 of the *moving* plane with respect to the unknown centerpoint M in the *fixed* plane.

As the body goes from its first to its j^{th} position, the circlepoint goes from K_1 to K_j, rotating about point M. The vector **Z** is rigidly attached to the moving plane, so it rides along with it from $\mathbf{Z}_1$ to $\mathbf{Z}_j$ and it rotates by the displacement angle ϕj.

Thus we have

Given

$$\mathbf{Z}_j = e^{i\phi_j}\,\mathbf{Z}_1$$

Unknown Unknown

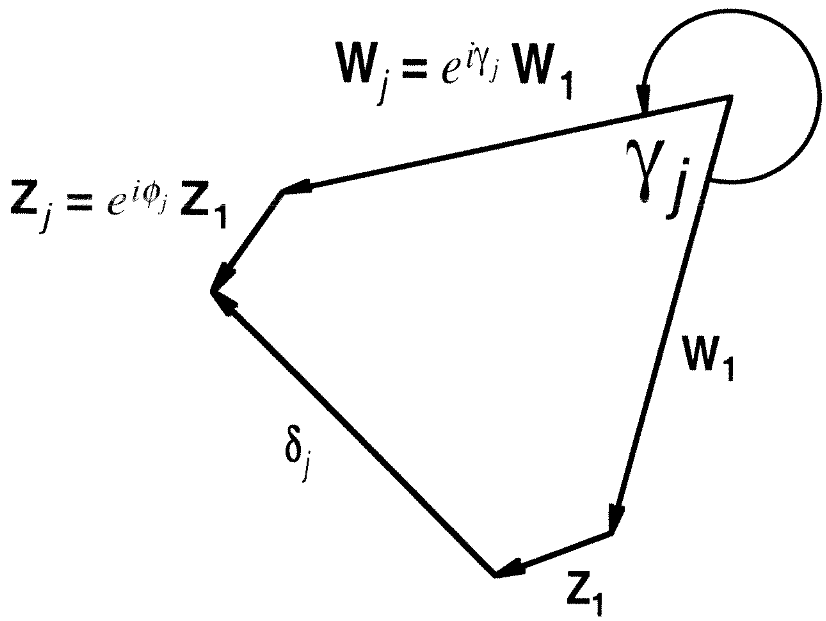
Meanwhile, back at the ranch, the unknown vector $\mathbf{W}_1$ has rotated through an unknown angle γ_j as $\mathbf{W}_1$ went to $\mathbf{W}_j$:

$$\underbrace{\mathbf{W}_j = e^{i\gamma_j}\,\mathbf{W}_1}_{\text{All Unknown}}$$

Point A has been displaced from A1 to Aj by the readily calculated amount

$$\boldsymbol{\delta}_j = \mathbf{r}_j - \mathbf{r}_1$$

Looking at the vector polygon consisting of $\boldsymbol{\delta}_j$, $\mathbf{Z}_j$, $\mathbf{Z}_1$, $\mathbf{W}_j$, and $\mathbf{W}_1$ we have



$$\boldsymbol{\delta}_j = \mathbf{Z}_j - \mathbf{Z}_1 + \mathbf{W}_j - \mathbf{W}_1$$

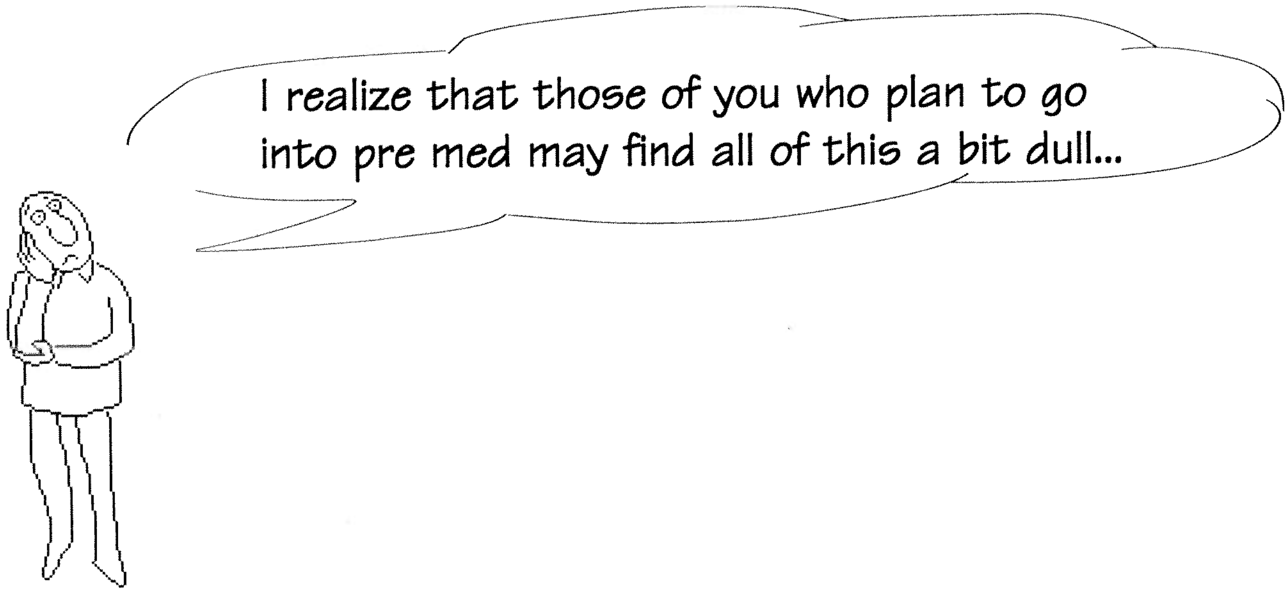
Equation of Closure

or

$$\boldsymbol{\delta}_j = \left(e^{i\phi_j} - 1\right)\mathbf{Z}_1 + \left(e^{i\gamma_j} - 1\right)\mathbf{W}_1$$

Loop Displacement Equation

A loop displacement equation can be written for *each* displaced position. ♫ **Note**: ♫ It describes the *unknown* vectors in terms of their *reference* configuration!



Three Specified Design Positions

For *three* given design positions you can write *two* loop closure equations as shown below:

$$\begin{bmatrix} (e^{i\phi_2} - 1) & (e^{i\gamma_2} - 1) \\ (e^{i\phi_3} - 1) & (e^{i\gamma_3} - 1) \end{bmatrix} \begin{bmatrix} \mathbf{Z}_1 \\ \mathbf{W}_1 \end{bmatrix} = \begin{bmatrix} \delta_2 \\ \delta_3 \end{bmatrix}$$

Given prescribed input data

? ? ? Real γ_j's

Two complex unknowns

Known from input data

This is a system of two complex non-homogeneous equations in $\mathbf{Z}_1$ and $\mathbf{W}_1$. You can arbitrarily pick values for γ_2 and γ_3 to meet your little heart's desires for the rotation of W and then solve to find the corresponding values for $\mathbf{Z}_1$ and $\mathbf{W}_1$.

Once you have solved for $\mathbf{Z}_1$ and $\mathbf{W}_1$ you can then solve for the circlepoints and centerpoints as

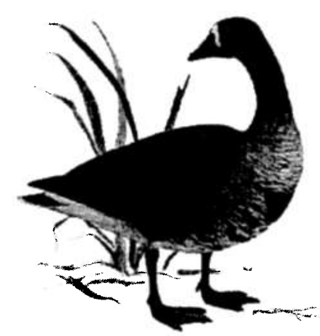
$$\mathbf{K}_1 = \mathbf{r}_1 - \mathbf{Z}_1$$

and

$$\mathbf{M} = \mathbf{K}_1 - \mathbf{W}_1$$

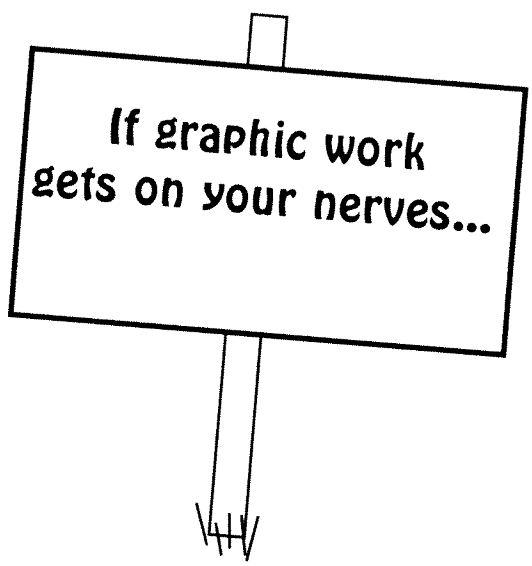
Varying the freely chosen values for γ_2 and γ_3 yields a double infinity of three position solution possibilities. This confirms what we already know, that is, *any* point is a circlepoint for three positions!

But take another gander at what is happening.

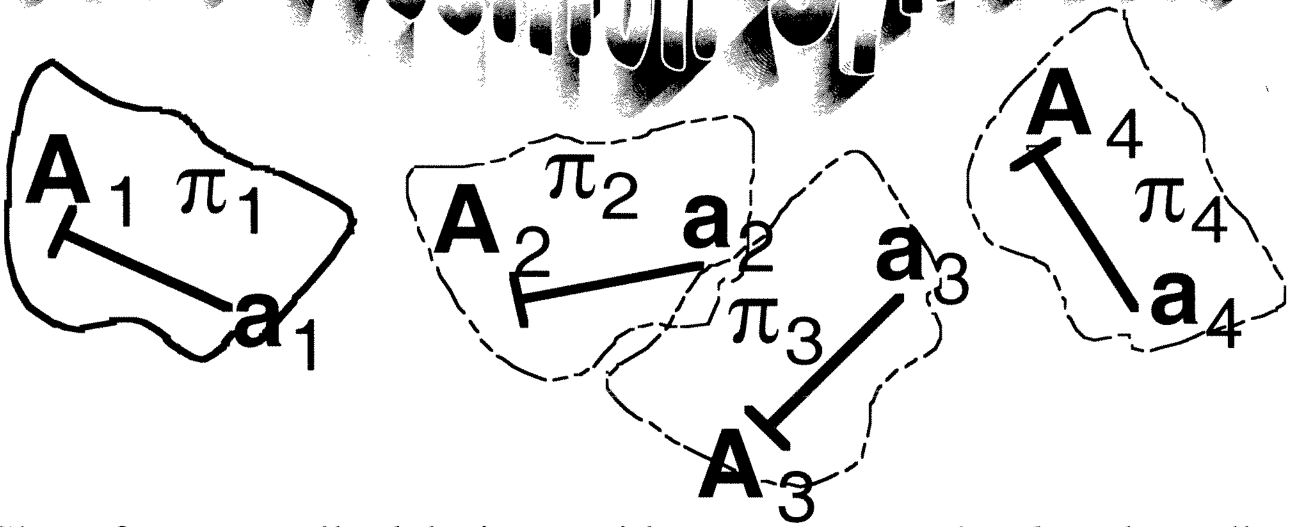


Assuming we are planning to use these circlepoints and centerpoints to design linkages like four-bars or slider-crank mechanisms, then γ_2 is the amount that the crank of the linkage will swing by as the body moves from position one to two, and γ_3 is the amount it rotates as the body moves from position one to three.

Since we can freely pick the gammas, we can actually *specify* the rotations of both cranks of our four-bar, for example, at the same time we are controlling the corresponding motion of the coupler body! In other words, we can specify the *timing* with which the body motion will occur, and we can simultaneously generate almost any desired functional relationship between the input and output shaft rotations with three accuracy point approximation! So we can synthesize a four-bar to correlate the motion of the coupler link (x and y displacement of the point A along with the rotations ϕ_j of the coupler) and at the same time specify the angular displacements of the input crank *and* the angular displacements of the output link!!!! Amazing. All this for three positions. Truly amazing.

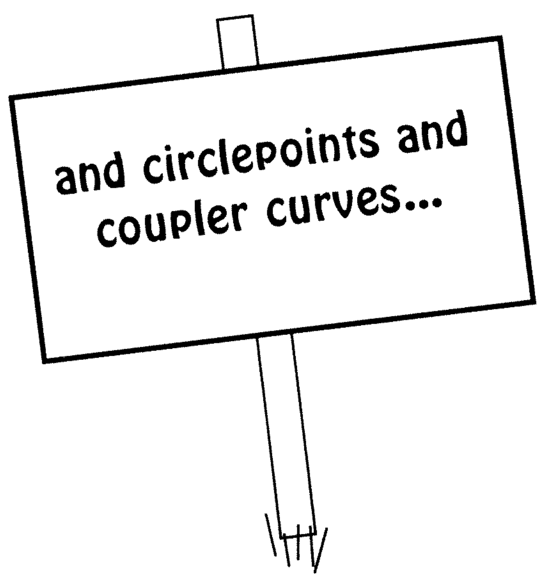


Four Position Synthesis

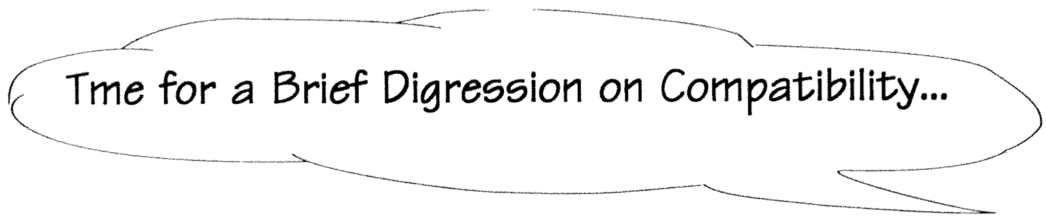


Given four prescribed design positions, one can write *three* loop displacement equations:

$$\begin{bmatrix} (e^{i\phi_2}-1) & (e^{i\gamma_2}-1) \\ (e^{i\phi_3}-1) & (e^{i\gamma_3}-1) \\ (e^{i\phi_4}-1) & (e^{i\gamma_4}-1) \end{bmatrix} \begin{bmatrix} \mathbf{Z}_1 \\ \mathbf{W}_1 \end{bmatrix} = \begin{bmatrix} \delta_2 \\ \delta_3 \\ \delta_4 \end{bmatrix}$$

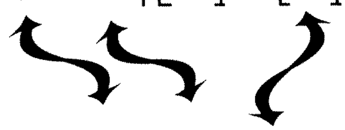


Since there are *three* complex equations and only *two* complex unknowns ($\mathbf{Z}_1$ and $\mathbf{W}_1$) the equations will only have a solution (or solutions) if the equations are linearly dependent, that is, if the coefficients satisfy certain "compatibility conditions."



Suppose we have the equations

$$\begin{bmatrix} 2 & -3 \\ -10 & 15 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$$



$$A = \left| \begin{array}{cc:c} 2 & -3 & 5 \\ -10 & 15 & 8 \end{array} \right|$$

The "Augmented Matrix"

$$M = \begin{bmatrix} 2 & -3 \\ -10 & 15 \end{bmatrix}$$

The "Coefficient Matrix"

Look at the left side of these equations:

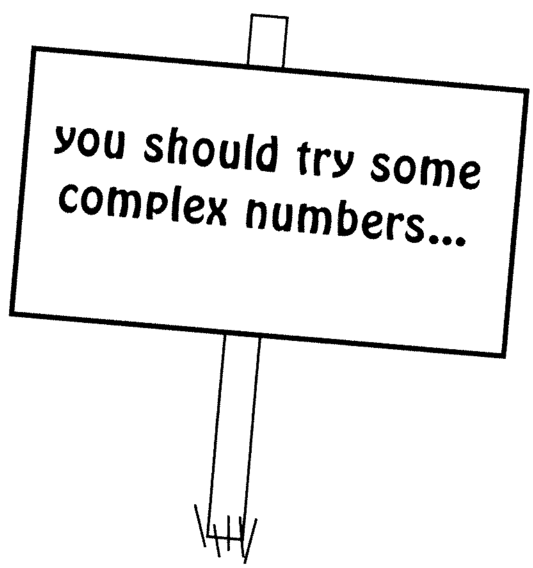
$$\begin{array}{r} 2X - 3Y \\ -10X + 15Y \end{array}$$

$\swarrow -5(2X - 3Y)$

The left side of the second is -5 times the left side of the first, so the *left* sides are linearly dependent.

Suppose we multiply the first equation through by -5. We get

$$-10X + 15Y = -25$$



but the second equation says

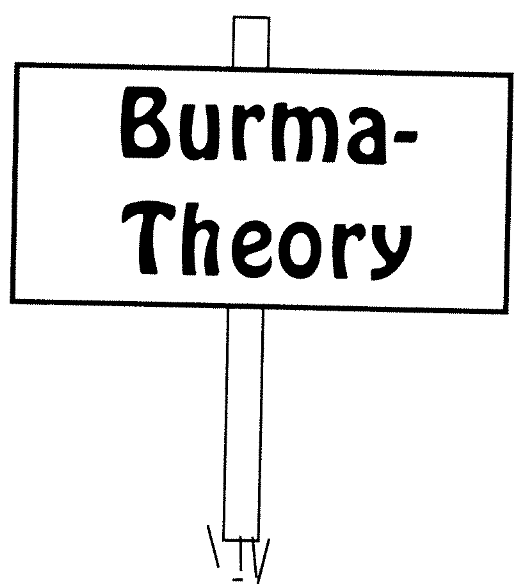
$$-10\,X + 15\,Y = 8$$

so the equations are *inconsistent*!

Now look at the matrix M:

$$M = \begin{bmatrix} 2 & -3 \\ -10 & 15 \end{bmatrix}$$

$$|M| = 0$$



Since the rows of M are *proportional*, $|M| = 0$. The *rank* of M is the order of the largest nonzero determinant contained in M. Thus, in this case, the rank of M is 1.

(This is getting to be a longer digression than I planned. Take Alka Seltzer for your indigression.)

Now consider the rank of the augmented matrix A. To find this, look at the values of the three two by two subdeterminants contained in A:

$$\begin{vmatrix} 2 & -3 \\ -10 & 15 \end{vmatrix} = 0$$

$$\left.\begin{aligned} \begin{vmatrix} 2 & 5 \\ -10 & 8 \end{vmatrix} &= 66 \\ \begin{vmatrix} -3 & 5 \\ 15 & 8 \end{vmatrix} &= -99 \end{aligned}\right\} \neq 0$$

Since at least one two by two determinant is nonzero, the rank of the augmented matrix A is *two.*

Rank A > Rank M

(and the equations are *inconsistent*!)

Suppose the equations had been

$$\begin{bmatrix} 2 & -3 \\ -10 & 15 \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 5 \\ -25 \end{bmatrix}$$

= -5 X 5

Now the right hand sides have the *same* ratio as the left hand sides. The equations are *consistent* and the

Rank A = Rank M = 1

Since the equations are proportional, we (the cognoscente) call them *dependent* and we might just as well have a single equation. For example, we could assign any value to X and solve either one of the equations for a corresponding Y.

For instance, the first equation gives

$$Y = \frac{2X - 5}{3}$$

We could just as well solve for X as Y.

The fact that we had *two* equations but

Rank M = 1

meant that only *one* of the two equations was linearly independent.

I assume there is a point to all this...

Thus, in order to be able to solve the equations, they *must be consistent*

Rank A = Rank M

(Since M is a submatrix of A, Rank A can never be less than Rank M)

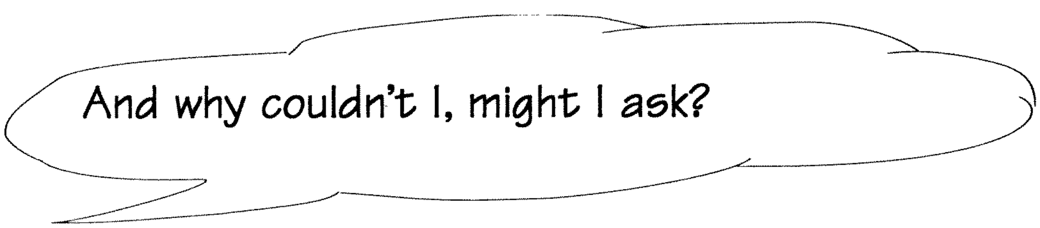
If Rank A = Rank M = n, the number of unknowns, then the equations are consistent and linearly independent. In that case, they can be solved for *one* solution for the n unknowns. If there are more equations than unknowns, however, you must be careful to choose n equations which are *linearly independent* or you could get blown away!

If you have n linearly independent equations you can solve them by any of the usual methods for the the n unknowns. A simple method that works well for two or three unknowns is using Cramer's Rule.

A z z a f o r i n s t a n c e , if you had the equations

$$\begin{bmatrix} 1 & 1 \\ 1 & -3 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 10 \end{bmatrix}$$

you could solve the first and second for X and Y or you could solve the second and third for X and Y but you couldn't solve using the first and third.



You couldn't because the first and third equations are *linearly related*. If you insist on trying despite that fact, then I would need to revise this to read that you couldn't solve them for the more fundamental reason that you appear to be dumber than a brick.

Finally, if Rank A = Rank M = r, but r < n (the number of unknowns) then there are an *infinite* number of solutions. (Maybe even more, but we want to keep this from becoming too esoteric a discussion.)

One can select r independent equations in r unknowns and solve these for the r unknowns in terms of the remaining n - r unknowns. You must solve for unknowns corresponding to one of the non-zero determinants of rank r. For instance, if you had the equations

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$$

then Rank M = Rank A = 2, since

$$\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} \neq 0$$

but you can't just set Z = 0 and try to solve for X and Y since the resulting equations are

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$$

which are *inconsistent*. You need to solve for unknowns corresponding to one of the non-zero 2 x 2 determinants, such as

$$\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}$$

Thus, you can solve for Y and Z in terms of X but not X and Y in terms of Z.

Meanwhile back at four-position Burmester Theory, we left our heroine with the following evil system of equations:

$$\begin{bmatrix} \left(e^{i\phi_2}-1\right) & \left(e^{i\gamma_2}-1\right) \\ \left(e^{i\phi_3}-1\right) & \left(e^{i\gamma_3}-1\right) \\ \left(e^{i\phi_4}-1\right) & \left(e^{i\gamma_4}-1\right) \end{bmatrix} \begin{bmatrix} \mathbf{Z}_1 \\ \mathbf{W}_1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{\delta}_2 \\ \boldsymbol{\delta}_3 \\ \boldsymbol{\delta}_4 \end{bmatrix}$$

Given prescribed input data — ? ? ? Real γ_j's — Two complex unknowns — Known from input data

These can only have a solution if the Rank A = Rank M = 2. That is,

$$\left[\begin{array}{cc:c} \left(e^{i\phi_2}-1\right) & \left(e^{i\gamma_2}-1\right) & \boldsymbol{\delta}_2 \\ \left(e^{i\phi_3}-1\right) & \left(e^{i\gamma_3}-1\right) & \boldsymbol{\delta}_3 \\ \left(e^{i\phi_4}-1\right) & \left(e^{i\gamma_4}-1\right) & \boldsymbol{\delta} \end{array}\right], \text{Rank } 2$$

<u>Compatibility Condition</u>

This condition can be expressed in the following compatibility equation:

$$\begin{vmatrix} \left(e^{i\phi_2}-1\right) & \left(e^{i\gamma_2}-1\right) & \boldsymbol{\delta}_2 \\ \left(e^{i\phi_3}-1\right) & \left(e^{i\gamma_3}-1\right) & \boldsymbol{\delta}_3 \\ \left(e^{i\phi_4}-1\right) & \left(e^{i\gamma_4}-1\right) & \boldsymbol{\delta}_4 \end{vmatrix} = 0$$

This is a complex determinant, so it can be solved for two real quantities or one complex number. Since the ϕ_j's and the δ_j's are specified, we must solve it for two of the γ_j's. We can arbitrarily pick γ_2, for instance, and then solve for γ_3 and γ_4.

Expanding the compatibility equation in terms of cofactors of the second column, we get

$$-\left(e^{i\gamma_2}-1\right)\underbrace{\begin{vmatrix}\left(e^{i\phi_3}-1\right) & \boldsymbol{\delta}_3\\ \left(e^{i\phi_4}-1\right) & \boldsymbol{\delta}_4\end{vmatrix}}_{-\Delta_2}+\left(e^{i\gamma_3}-1\right)\underbrace{\begin{vmatrix}\left(e^{i\phi_2}-1\right) & \boldsymbol{\delta}_2\\ \left(e^{i\phi_4}-1\right) & \boldsymbol{\delta}_4\end{vmatrix}}_{\Delta_3}$$

$$-\left(e^{i\gamma_4}-1\right)\underbrace{\begin{vmatrix}\left(e^{i\phi_2}-1\right) & \boldsymbol{\delta}_2\\ \left(e^{i\phi_3}-1\right) & \boldsymbol{\delta}_3\end{vmatrix}}_{-\Delta_4}=0$$

or

$$e^{i\gamma_2}\Delta_2+e^{i\gamma_3}\Delta_3+e^{i\gamma_4}\Delta_4=\Delta_1$$

where

$$\Delta_1 = \Delta_2 + \Delta_3 + \Delta_4$$

and where all of the Δj's are 2X2 complex determinants, readily calculated from the given input data. Thus, the Δj's are known complex numbers.

As an alternate notation, let us write

$$\Delta_2 = R_2 e^{i\mu_2}$$
$$\Delta_3 = R_3 e^{i\mu_3}$$
$$\Delta_4 = R_4 e^{i\mu_4}$$

and

$$\Delta_1 = \sum_{j=2}^{4} \Delta_j = R_1 e^{i\mu_1}$$

Suppose we choose to solve for γ_3 and γ_4 in terms of γ_2. Then let

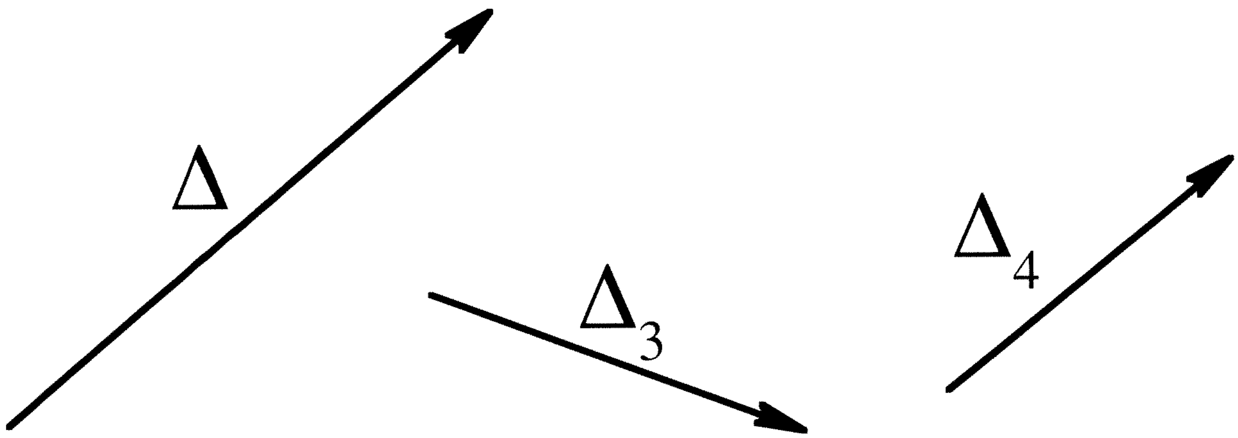
$$\Delta = \Delta_1 - e^{i\gamma_2}\Delta_2 = R\,e^{i\mu}$$

Without further embelishment, we get the following elegantly concise equation:

$$e^{i\gamma_3}\Delta_3 + e^{i\gamma_4}\Delta_4 = \Delta$$

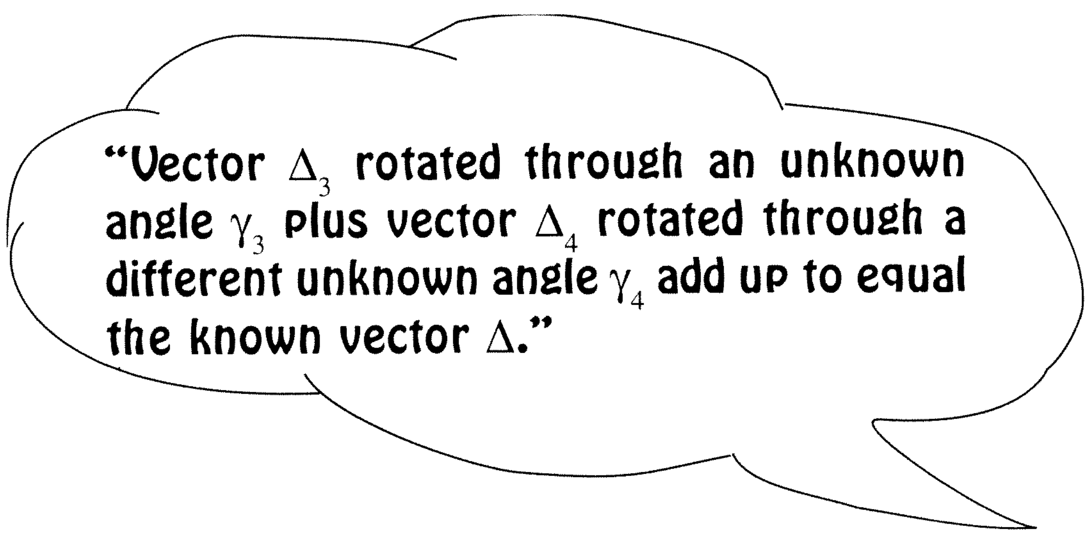
Each choice of a γ_2 results in a known Δ and a corresponding equation of this form which can be solved for corresponding compatible values for γ_3 and γ_4.

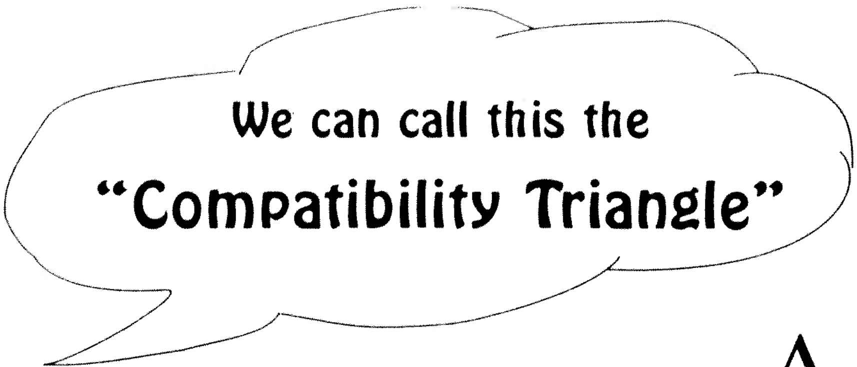
This modest equation can be given the following graphical interpretation:

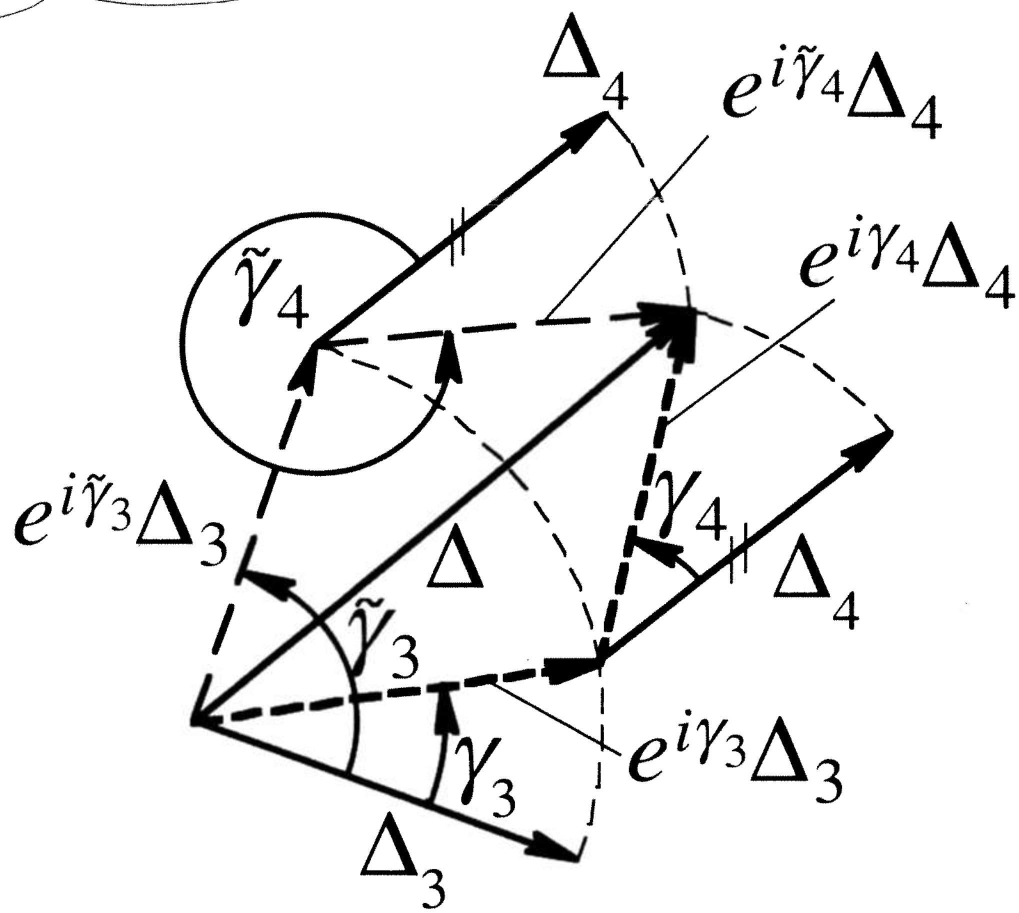


The complex numbers Δ, Δ_3, and Δ_4 are *known* complex numbers at this point and can be visualized as vectors as shown above.

How can we visualize $e^{i\gamma_3}\Delta_3$? $e^{i\gamma_3}\Delta_3$ simply differs from Δ_3 in that the vector has been rotated through some *unknown* angle γ_3. The modest little equation on the preceeding page is a *loop closure equation* for a vector triangle. It says (if it could speak):





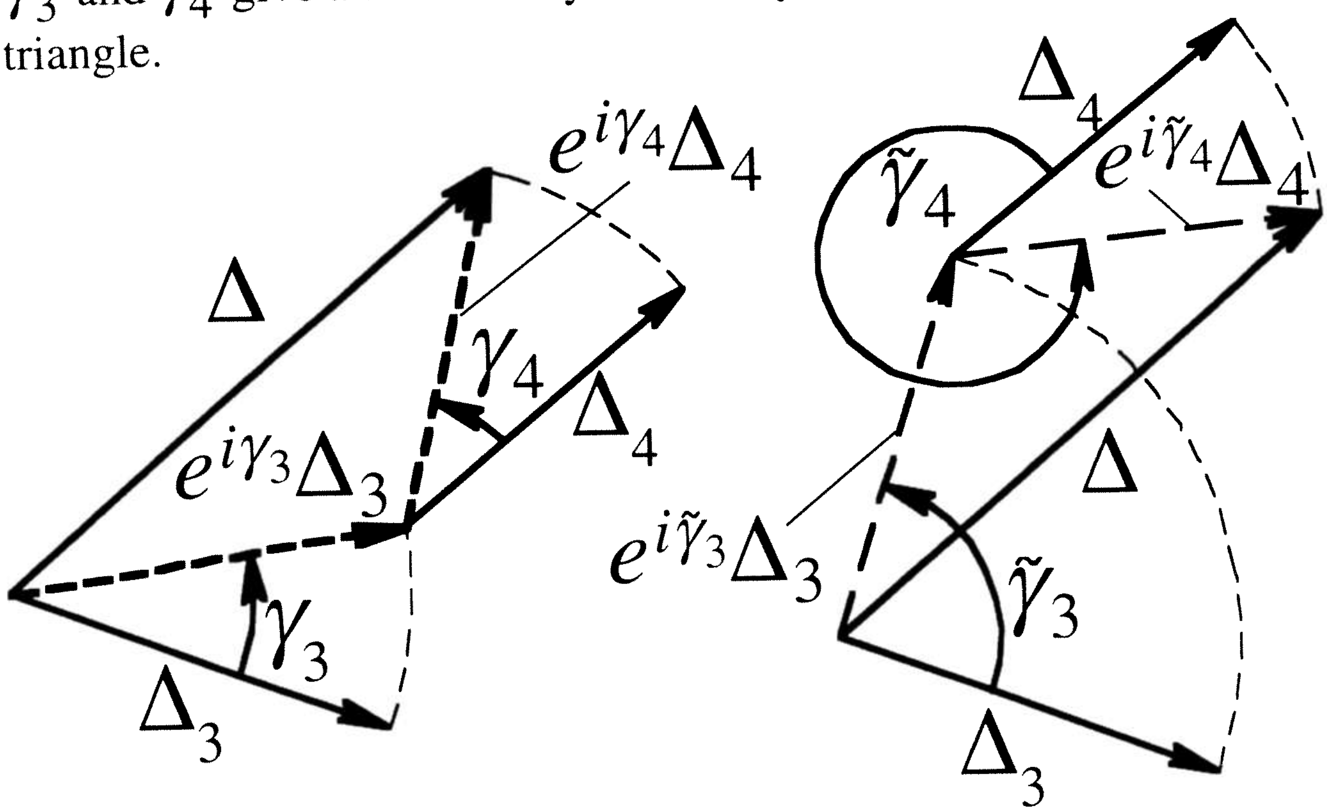


As can be seen from this figure, the compatible solutions for γ_3 and γ_4 are given by the two possible ways in which the vector triangle can be assembled. Clearly (or if you prefer, "It is intuitively obvious") (something that people say when they know damn well it is as obscure as mud) there are two possible ways to form a triangle given the fixed vector Δ and given the known lengths of the vectors $e^{i\gamma_3}\Delta_3$ and $e^{i\gamma_4}\Delta_4$.

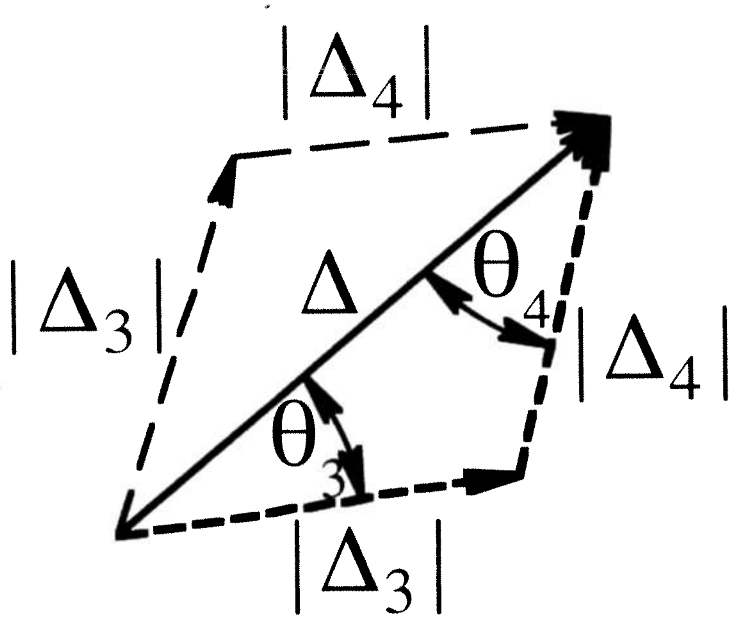
You could rotate the original known vector Δ_3 by the angle γ_3 as shown to get the vector $e^{i\gamma_3}\Delta_3$. Similarly, you could rotate the known vector Δ_4 by the angle γ_4 to get the vector $e^{i\gamma_4}\Delta_4$. Vectors $e^{i\gamma_3}\Delta_3$ plus $e^{i\gamma_4}\Delta_4$ add up to equal the known vector Δ.

Alternatively, you could rotate the original vector Δ_3 by the a *different* angle γ_3 — an angle we'll call $\tilde{\gamma}_3$ — to get the vector $e^{i\tilde{\gamma}_3}\Delta_3$. Similarly, you could rotate the known vector Δ_4 by an alternate value of angle γ_4 — we'll call this $\tilde{\gamma}_4$ — to get the vector $e^{i\tilde{\gamma}_4}\Delta_4$. Vectors $e^{i\tilde{\gamma}_3}\Delta_3$ plus $e^{i\tilde{\gamma}_4}\Delta_4$ *also* add up to equal the known vector Δ.

Rotations γ_3 and γ_4 allow one assembly of the triangle and rotations $\tilde{\gamma}_3$ and $\tilde{\gamma}_4$ give a second way in which you can assemble the vector triangle.



Algebraically, you can determine the magnitudes of the internal angles of the triangle by using the *Law of Cosines.* (Come on, now. Don't pretend you don't remember the Law of Cosines. That's the one you learned back in high school that says "If you have a triangle with sides of lengths a, b, and c opposite the angles α, β, and γ then $c^2 = a^2 + b^2 - 2\,a\,b\cos\gamma$) Let's use it to determine the interior angles θ_3 and θ_4 of this vector triangle:



From the figure above we can see that

$$\theta_3 = \cos^{-1}\left(\frac{|\Delta^2| + |\Delta_3^2| - |\Delta_4^2|}{2\,\Delta\,\Delta 3}\right)$$

$$\theta_4 = \cos^{-1}\left(\frac{|\Delta^2| + |\Delta_4^2| - |\Delta_3^2|}{2\,\Delta\,\Delta}\right)$$

A few pages back (when this discussion was literally on track) I introduced the shorthand notation of using μ, μ_3, and μ_4 to stand for the known slopes of the vectors Δ, Δ_3, and Δ_4.

With that notation, the solution for the unknown angles γ_3, γ_4, $\tilde{\gamma}_3$, and $\tilde{\gamma}_4$ becomes:

$$\left.\begin{aligned} \gamma_3 &= \mu - \mu_3 - \theta_3 \\ \gamma_4 &= \mu - \mu_4 + \theta_4 \end{aligned}\right\} \textit{Both of these angles are compatible with the chosen } \gamma_2$$

$$\left.\begin{aligned} \tilde{\gamma}_3 &= \mu - \mu_3 + \theta_3 \\ \tilde{\gamma}_4 &= \mu - \mu_4 - \theta_4 \end{aligned}\right\} \textit{These angles are also compatible with the chosen } \gamma_2$$

(At this point you should stop and check to see if vectors $e^{i\gamma_3}\Delta_3$ plus $e^{i\gamma_4}\Delta_4$ add up to equal the known vector Δ. If they don't, the darned arrow head was on the other end of either the Δ_3 or the Δ_4 vector. To fix this minor snafu you just need to add pi to the value of θ_4 in the equations above and recalculate γ_4 and $\tilde{\gamma}_4$.)

Now that we have found compatible sets of values for γ_2, γ_3, and γ_4, we can go back and solve any two of our original equations for $\mathbf{Z}_1$ and $\mathbf{W}_1$ just the way we did for three positions. Just as a reminder,

$$\begin{bmatrix} (e^{i\phi_2}-1) & (e^{i\gamma_2}-1) \\ (e^{i\phi_3}-1) & (e^{i\gamma_3}-1) \end{bmatrix} \begin{bmatrix} \mathbf{Z}_1 \\ \mathbf{W}_1 \end{bmatrix} = \begin{bmatrix} \delta_2 \\ \delta_3 \end{bmatrix}$$

You can use any of the standard methods to solve these equations. For instance, using Cramer's Rule (as opposed to the Peter Principle or Murphy's Law):

$$\mathbf{Z}_1 = \frac{\begin{vmatrix} \delta_2 & (e^{i\gamma_2}-1) \\ \delta_3 & (e^{i\gamma_3}-1) \end{vmatrix}}{\begin{vmatrix} (e^{i\phi_2}-1) & (e^{i\gamma_2}-1) \\ (e^{i\phi_3}-1) & (e^{i\gamma_3}-1) \end{vmatrix}}$$

$$\mathbf{W}_1 = \frac{\begin{vmatrix} (e^{i\phi_2}-1) & \delta_2 \\ (e^{i\phi_3}-1) & \delta_3 \end{vmatrix}}{\begin{vmatrix} (e^{i\phi_2}-1) & (e^{i\gamma_2}-1) \\ & \end{vmatrix}}$$

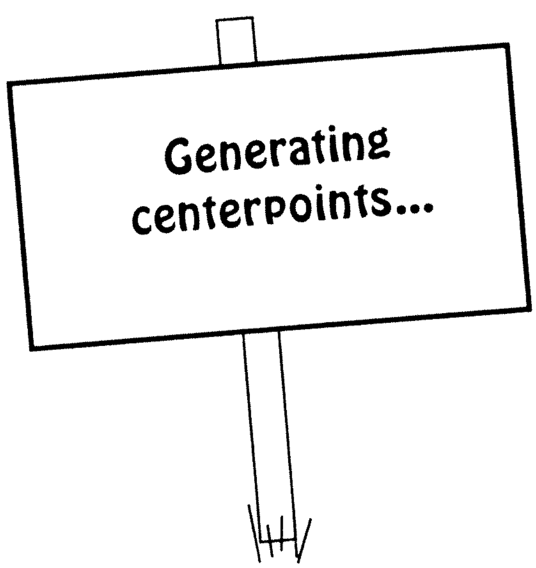
Once you have solved for $\mathbf{Z}_1$ and $\mathbf{W}_1$ you can then solve for the circlepoints and centerpoints as

$$\mathbf{K}_1 = \mathbf{r}_1 - \mathbf{Z}_1$$

and

$$\mathbf{M} = \mathbf{K}_1 - \mathbf{W}_1$$

Remember, point $\mathbf{K}_1$ is a point on the moving body as seen in it's reference (#1) position. Point $\mathbf{M}$ is a corresponding point on the fixed body. Point $\mathbf{K}_1$ circles around point $\mathbf{M}$ as the body moves through the four positions. Since points $\mathbf{K}_1$ and $\mathbf{M}$ correspond to one another they are sometimes called a "Circlepoint-Centerpoint Pair".



Using the alternate compatible set of values for γ_2, $\tilde{\gamma}_3$, and $\tilde{\gamma}_4$ in these same equations gives an alternate solution for vectors $\mathbf{Z}_1$ and $\mathbf{W}_1$ (we can call these $\tilde{\mathbf{Z}}_1$ and $\tilde{\mathbf{W}}_1$). Using them in the above equations we can get another circlepoint-centerpoint pair that is also compatible with the same crank rotation angle γ_2. We can call these points $\tilde{\mathbf{K}}_1$ and $\tilde{\mathbf{M}}$.

Thus, for each arbitrary choice we make for the angle γ_2 we obtain two circlepoints and two centerpoints. Wiggling the angle γ_2 slightly yields a slight change in the locations of these circlepoints and centerpoints.

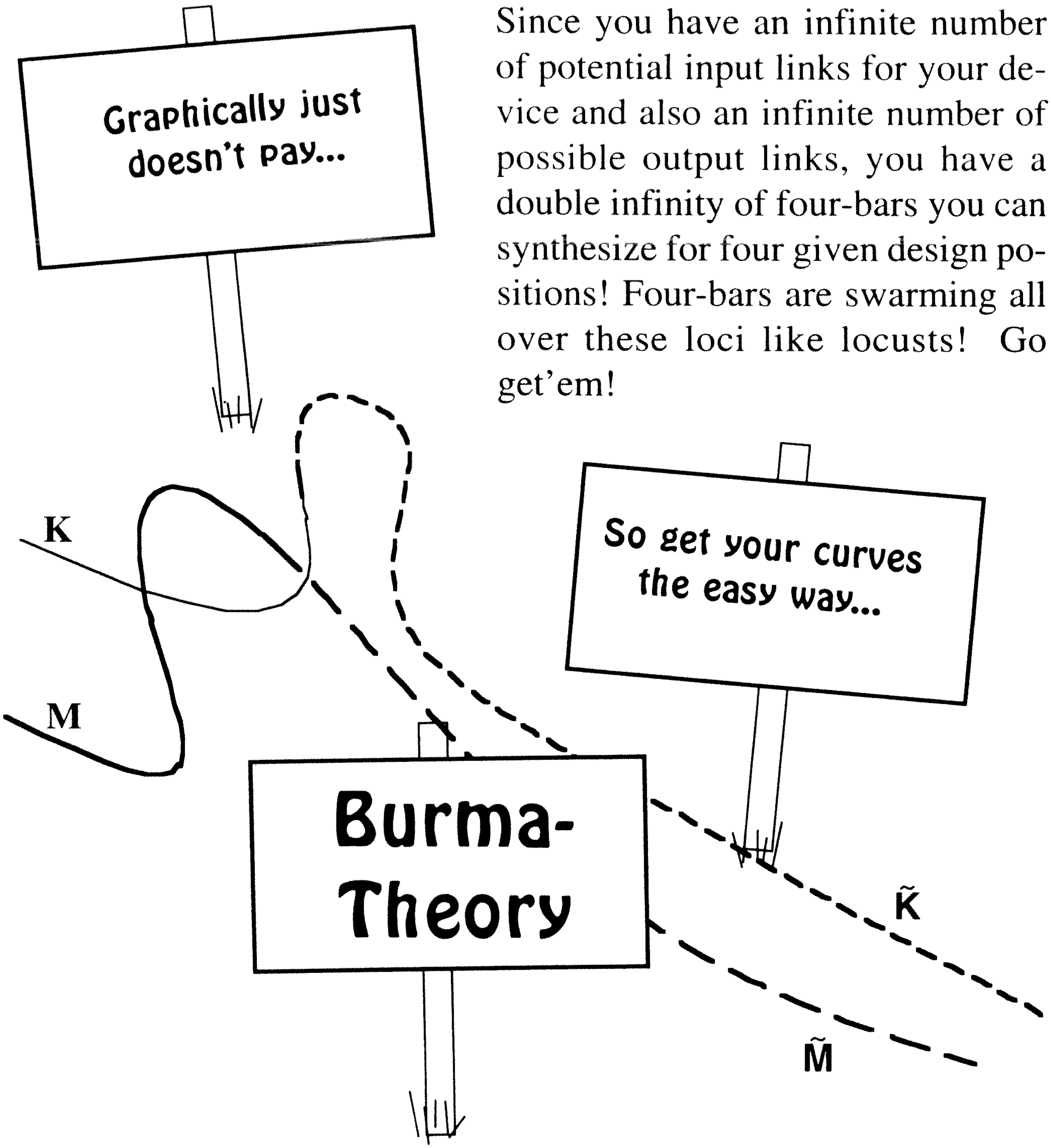
Sweeping the angle γ_2 around in a complete circle from $0 \le \gamma_2 < 2\pi$ causes points $\mathbf{K}_1$ and $\tilde{\mathbf{K}}_1$ to each sweep out a branch of the "Circlepoint Curve" or locus. At the same time, points **M** and $\tilde{\mathbf{M}}$ will generate branches of the "Centerpoint Locus."

There is a one-to-one mapping or correspondence between the points on the Circlepoint Locus and the points on the Centerpoint Locus. There are an infinite number of points on these two curves, with two pairs of points corresponding to each value of γ_2 . Thus the possible choices for four-bar mechanism links is infinite for four positions.

Pick off of the loci any two circlepoint-centerpoint pairs that strike your fancy and you have yourself a potential four-bar. (Just be careful you don't get your fancy caught in the four-bar. Pick one pair for the driver link and another pair for the follower. You can pick either of these links off of either branch of the locus you like. The two

branches are functionally identical. They just happened to pop out that way because of the way in which we derived the equations.)

Since you have an infinite number of potential input links for your device and also an infinite number of possible output links, you have a double infinity of four-bars you can synthesize for four given design positions! Four-bars are swarming all over these loci like locusts! Go get'em!



Notice that varying the angle γ_2 varies the side Δ of the "com patibility triangle." For certain magnitudes of Δ, the tri angle will flatten out, either with

$$|\Delta| = |\Delta_3| + |\Delta_4|$$

or with

$$|\Delta| = \left| |\Delta_3| - |\Delta_4| \right| .$$

When this happens, the loci $\mathbf{K}_1$ and $\tilde{\mathbf{K}}_1$ join together as do the curves $\mathbf{M}$ and $\tilde{\mathbf{M}}$.

For certain values of γ_2 it may not be possible to assemble the "compatibility triangle" at all. In this case, a mathematician would say the points $\mathbf{K}_1$, $\tilde{\mathbf{K}}_1$, $\mathbf{M}$ and $\tilde{\mathbf{M}}$ still had solutions but that the solutions were all imaginary. The rest of us would probably say that the mathematician was seeing things and should be locked up someplace where he won't hurt himself or us real-world engineers.

Algorithm for Burmester Curve Determination à la Sandor, Freudenstein, & Their Progeny